

Scaling for the generalized Ohm's law

Let's look at the magnitude of each term in the generalized Ohm's law to get a sense of which terms (in SSX say) dominate. Often in a complex expression like Ohm's law, a few terms will dominate so solving it (or even just understanding it) is easier.

The generalized Ohm's law comes essentially from the momentum equation of motion for the electron fluid. Its derivation can be found in some plasma physics books (like Goldston). You would consider separately the equations of motion for ions and electrons noting that $J \propto v_i - v_e$. It can be written (in MKS units):

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} + \frac{1}{ne} \mathbf{J} \times \mathbf{B} - \frac{1}{ne} \nabla P + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t}$$

Since this is an MKS formula, every term is an electric field with units of volts/meter. If you multiply through by a charge, then every term becomes a force. The left hand side is the electric field in the moving frame (comes from relativity). The first term on the right hand side comes from the electron drag on the ions (called the resistive term). The second term is the Hall term and has to do with the idea that electrons and ions can decouple and move separately. The third term comes from the force due to gradients in electron pressure. The last term comes from considering electron inertia (ie the ma part of $F = ma$). There are other terms (related to ion inertia and ion pressure for example), but they are smaller than those listed above. In general, the pressure can be a tensor so the pressure term is really the div of a pressure tensor. We'll stick with scalar pressure here.

The first step is to write the equation in dimensionless form. This is really important for doing numerical modeling. Computers don't like dealing with big numbers like our particle density (10^{21} in MKS) at the same time as small numbers like the electron mass (10^{-31} in MKS). Dimensionalizing an equation also gives a very good intuitive sense of what the equation means. It also makes it easier to compare different systems (say SSX and the solar corona).

Instead of MKS units, let's consider units more appropriate for SSX. Instead of meters, we'll use a more typical scale length for the problem: $\ell = 0.1m$. We'll re-write every length $x \rightarrow \ell \tilde{x}$ where x was a length measured in meters, \tilde{x} is a dimensionless number, and $\ell = 0.1m$. This means that if $\tilde{x} = 1$ we're talking about a length of 0.1 meters.

We do that for all the dynamical variables:

$$v \rightarrow v_A \tilde{v}$$

$$B \rightarrow B_0 \tilde{B}$$

$$E \rightarrow E_0 \tilde{E} = v_A B_0 \tilde{E}$$

So for example, if $\tilde{v} = 0.1$ that means a velocity 1/10 the Alfvén velocity where $v_A = 10^4 m/s$ for us. Similarly, $B_0 = 0.1T$ for us so $E_0 = 10^3 V/m$ is our electric field unit.

$$J = \frac{\nabla \times B}{\mu_0} \rightarrow \frac{B_0}{\ell \mu_0} \tilde{j}$$

Note that here in Ampere's law, the gradient operator (which normally has units of an inverse length) has also been rendered dimensionless.

$$t \rightarrow \tau_A \tilde{t} = \frac{\ell}{v_A} \tilde{t}$$

We'll need a few formulas later:

$$v_A = \frac{B_0}{\sqrt{\mu_0 n M}} \quad S \equiv \frac{\mu_0 v_A \ell}{\eta}$$

$$\omega_{pi}^2 = \frac{ne^2}{M\epsilon_0} \quad \delta_i^2 = \frac{c^2}{\omega_{pi}^2} = \frac{1}{\mu_0 \epsilon_0} \frac{M\epsilon_0}{ne^2} = \frac{M}{\mu_0 ne^2}$$

There's a similar expression for the electron inertial length δ_e .

$$\beta = \frac{W_K}{W_B} = \frac{nkT}{B^2/\mu_0}$$

This is just the kinetic pressure scaled to the magnetic pressure (instead of using Pascals for example).

$$\frac{\delta_i}{\ell} = \frac{1}{e\ell} \sqrt{\frac{M}{\mu_0 n}} = \frac{1}{\omega_{ci} \tau_A}$$

This last expression is interesting. It says that the number of ion inertial scales in SSX is the same as the number of gyro-orbits an ion can execute in an Alfvén time.

Dimensionless equation: First thing to do is divide the whole equation by $E_0 = v_A B_0$ so that all the electric field terms will be expressed in units of our maximum expected electric field (about 1000 V/m). We get:

$$\tilde{E} + \tilde{v} \times \tilde{B} = \frac{1}{v_A B_0} \left[\frac{\eta B_0}{\ell \mu_0} \tilde{J} + \frac{1}{ne} \frac{B_0^2}{\ell \mu_0} \tilde{J} \times \tilde{B} - \frac{1}{ne} \frac{P_0}{\ell} \tilde{\nabla} \tilde{P} + \dots \right]$$

Resistive term: The first term on the right (resistive term) becomes:

$$\frac{\eta}{\mu_0 v_A \ell} \tilde{J} = \frac{1}{S} \tilde{J}$$

so the resistive term scales like the inverse Lundquist number (about 1/1000 for us).

Hall term: The second term looks like:

$$\frac{B_0}{v_A n e \ell \mu_0}$$

We recall from above the expressions for the Alfvén speed and $\frac{\delta_i}{\ell}$ and we find that:

$$\frac{B_0}{v_A n e \ell \mu_0} = \frac{\sqrt{\mu_0 n M}}{n e \ell \mu_0} = \frac{1}{e \ell} \sqrt{\frac{M}{\mu_0 n}} = \frac{\delta_i}{\ell}$$

So the dimensionless Hall term becomes:

$$\frac{\delta_i}{\ell} \tilde{J} \times \tilde{B}$$

Electron pressure term: The third term is very similar to the Hall term. We need to recall the definition of β from above:

$$P_0 = \beta B_0^2 / \mu_0$$

So the electron pressure term looks like:

$$\beta \frac{B_0}{v_A n e \ell \mu_0} \tilde{\nabla} \tilde{P} = \beta \frac{\delta_i}{\ell} \tilde{\nabla} \tilde{P}$$

Electron inertia term: Finally, using our expressions for dimensionless current and dimensionless time, the last term becomes:

$$\frac{1}{v_A B_0} \frac{m_e v_A B_0}{n e^2 \ell} \frac{\partial \tilde{J}}{\partial \tilde{t}} = \left(\frac{\delta_e}{\ell} \right)^2 \frac{\partial \tilde{J}}{\partial \tilde{t}}$$

Dimensionless Ohm's law: Let's drop the twiddle and write down the full dimensionless Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{S} \mathbf{J} + \frac{\delta_i}{\ell} \mathbf{J} \times \mathbf{B} - \beta \frac{\delta_i}{\ell} \nabla \mathbf{P} + \left(\frac{\delta_e}{\ell} \right)^2 \frac{\partial \mathbf{J}}{\partial t}$$

Some numbers: Roughly speaking for SSX, $S = 1000$, $\delta_i = 1cm$, $\delta_e = 1/4mm$, $\ell = 10cm$ and $\beta = 0.1$. So the scale of the terms on the right hand side of Ohm's law is about 0.001, 0.1, 0.01, 0.00005 for resistive, Hall, electron pressure, and electron inertia, respectively in SSX. Hall wins, and that's what we measure.

For the corona, $S = 10^8$, $\delta_i = 1m$, $\delta_e = 2cm$, $\ell = 10,000km$ and $\beta = 0.01$. So the scale of the terms on the right hand side of Ohm's law is about 10^{-8} , 10^{-7} , 10^{-9} , 10^{-10} . The solar corona is pretty ideal... ie $E + v \times B \cong 0$ to high accuracy, yet reconnection is pretty fast there too.

Other random relations: Other useful stuff to know...

$$\omega_{ci} = \frac{eB_0}{M} \quad \rho_i = \frac{v_i}{\omega_{ci}}$$

$$\left(\frac{\rho_i}{\delta_i} \right)^2 = \beta \quad \text{and} \quad \left(\frac{v_i}{v_A} \right)^2 = \beta$$

$$\text{so } \rho_i v_A \cong \delta_i v_i$$

$$\lambda_D^2 = \frac{\epsilon_0 kT}{n e^2} \quad \left(\frac{\lambda_D}{\delta_e} \right)^2 = \frac{\epsilon_0 kT}{n e^2} \frac{n e^2}{c^2 m_e \epsilon_0} = \frac{kT}{m_e c^2}$$

$$\frac{\eta S_p}{\mu_0} = \delta_e^2 \nu_{ei}$$