Some notes about β

The dimensionless number β is closely related to the square of the ratio of the ion gyro-radius to the ion inertial length. We have for the gyro-radius:

$$\rho_i = \frac{v_{ti}}{\omega_{ci}}$$

where $\omega_{ci} = eB/M$. The ion inertial length is:

$$\delta_i = \frac{c}{\omega_{pi}}$$

where $\omega_{pi} = \sqrt{me^2/M\epsilon_0}$.

In other notes, I've shown that $\omega_{ci}\delta_i = v_A$. This is because:

$$\omega_{ci}\delta_i = \frac{eB}{M}c\sqrt{\frac{M\epsilon_0}{ne^2}} = \frac{B}{\sqrt{nM\mu_0}} = v_A$$

where $c^2\epsilon_0\mu_0 = 1$ in MKS units.

Now, if I make the assumption that $T_e = T_i$, then $k(T_e + T_i) = 2kT = 2(Mv_i^2/2) = Mv_i^2$. Now,

$$\beta = \frac{nkT}{B^2/2\mu_0} = \frac{2nkT\mu_0}{B^2} = \frac{n\mu_0 M v_i^2}{B^2}$$

$$\left(\frac{\rho_i}{\delta_i}\right)^2 = \frac{v_i^2}{\omega_{ci}^2}\frac{\omega_{pi}^2}{c^2} = \frac{M^2v_i^2}{e^2B^2}\frac{ne^2}{M\epsilon_0}\frac{1}{c^2} = \frac{nMv_i^2}{B^2\epsilon_0 c^2}$$

The last two expressions are equal since $c^2\epsilon_0\mu_0 = 1$.

There is another connection (using the result above) since

$$\left(\frac{\rho_i}{\delta_i}\right) = \frac{v_i}{\omega_{ci}\delta_i} = \frac{v_i}{v_A} \rightarrow \beta = \left(\frac{\rho_i}{\delta_i}\right)^2 = \left(\frac{v_i}{v_A}\right)^2$$

Note also the "electron Alfvén speed":

$$\omega_{ce}\delta_e = \frac{eB}{m}c\sqrt{\frac{m\epsilon_0}{ne^2}} = \frac{B}{\sqrt{nm\mu_0}} = v_{Ae}$$