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A Dynamical System with a Z^2 Centralizer

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1. INTRODUCTION

In [PR], K. K. Park and E. A. Robinson construct a class of Z^2 actions and study the joinings within this class. These actions are the natural 2-dimensional analogs to the Chacon transformation and are called Chacon Z^2 Actions. They are produced by a 2-dimensional rank 1 cutting and stacking construction which is reviewed in Subsection 2.1. Some of the results in [PR] follow from the general theory of joinings but others depend on the specific algebraic and geometric properties of the group Z^2. In particular, the group Z^2 has Z for a nontrivial subgroup and the purpose of this paper is to study this particular subgroup action. Let the Chacon Z^2 action be denoted by $\{T^iS^j : (i, j) \in Z^2\}$. Then T and S each generate a Z-subaction. Ergodicity of the Chacon Z^2 action is clear from the construction but it is the ergodicity of the Z action that will be needed for this paper. This result is a consequence of the careful choice of pattern used in the Chacon Z^2 actions and is proven in [PR]. In this paper we will write statements in terms of T and leave the analogous statements and proofs for S to the reader. We conjecture that in fact they can be extended to $T^{i_o}S^{j_o}$ for any fixed (i_o, j_o). We divide this paper into two parts,

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corresponding to our two main results:

(1) Let P be the time zero partition, described further in Section 2. Then P is a generating partition under T.

(2) The centralizer of T is the Z^2 group $\{T^iS^j : (i, j) \in Z^2\}$.

Thus the second result yields a centralizer which is countable but nontrivial. There are not many actions with a countable centralizer. For instance, rotations on the circle automatically give an uncountable centralizer. The Ledrappier 3-dot example [S] is a Z^2 action with a Z-subaction which, although it does not generate as in (1), does exhibit a certain generating property. However, the centralizer of this subaction is uncountable. Also, J. King [K] has shown there is a rank 1 Z^2 action with a Bernoulli subaction, thus this subaction has an uncountable centralizer. On the other hand, D. Rudolph has shown in [R] that if U has minimal self joinings then the centralizer of $U \times U$ is $\{U^i \times U^j : (i, j) \in Z^2\} \cup (\phi : \phi(x, y) = (y, x)\}$. However, that example of a countable centralizer has different properties than the Chacon example explored in this paper. For instance, ʼhe centralizer of T has the property of minimal self joinings as a Z^2-action while the centralizer of $U \times U$ above does not.

The entropy of T can be shown to be zero simply by using the structure of the symbolic description of the action. Notice that if the weak Pinsker property were known to be universal, then this result would follow directly from (2). Also notice that (2) shows that T is a coalescent map, as defined, for example, in [GLL] by P. Gabriel, M. Lemanczyk, and P. Liardet.

Note that result (1) says that we do not need the entire Z^2-name in order to distinguish points. Without this result, any approximation of a map $\phi \in C(T)$ by a finite code would depend on rectangular names, as would a comparison of the two images. But this would require ϕ to also commute with S. Thus result (1) becomes a prerequisite of result (2). Result (2) is the natural extension of A. delJunco's result [dJ] on the centralizer of the 1-dimensional Chacon action. It is not possible to directly use his methods in the 2-dimensional case, because 2-dimensional cutting and stacking yields many possibilities for the blocks seen in one direction. Thus there are many cases that must be exhausted in the proof of (2). An alternate method one could use to find the centralizer of T is to first investigate the self joinings of T and show that the set of self joinings consists of measures concentrated on $\{(x, T^iS^jx) : (i, j) \in Z^2\}$. This can be done in a manner similar to that used by delJunco, Rahe, and Swanson in [dJRS] for the 1-dimensional Chacon action and in some ways is slightly simpler than the direct approach used in this paper. However, it still involves investigating many different cases and does not display the structure of T as well as the direct approach. The structure explored here may be useful in investigating the actions in other directions, including questions about their directional entropy. We also believe that our direct approach is useful in the study of substitution systems.

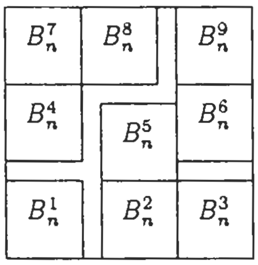
2. A 1-DIMENSIONAL GENERATOR FOR THE CHACON Z^2 ACTION

2.1. *Review of Construction*

The construction of the Chacon Z^2 action was done by K. K. Park and E. A. Robinson in [PR]. The prototypical example will be reviewed here. We leave the interested reader to investigate the general situation in [PR].

Let $[0,1)$ be cut into two pieces: $[0, 43/47)$ which is the "spacer" portion, and $[43/47, 1)$ which is further cut into nine equal pieces of length l_o labeled L^0_{ij} with $i, j = 0,1,2$. These are stacked into a 3×3 box, B_o, and left and up translation are defined where possible, i.e., $T : L^0_{ij} \to L^0_{(i+1)j}$, $S : L^0_{ij} \to L^0_{i(j+1)}$.

Let $h_o = 3$ and $h_n = 3h_{n-1} + 1$. Then the construction is done inductively by thinking of B_n as $(1/9)^n \cdot l_o \times [0, \dots, h_n - 1] \times [0, \dots, h_n - 1]$. So B_n is a $h_n \times h_n$ box with each "position" containing an interval of length $l_o/9_n$. These intervals are again cut into nine pieces. Group together the first ninth of each interval, maintaining their relative positions, and label it B^1_n. Similarly label the second ninth B^2_n, etc. Place these together as shown below to create B_{n+1} where the gaps are filled in with intervals of the same length, $l_o/9^{n+1}$, cut from the spacer portion of $[0,1)$. T and S are again left and up translation.

$B_{n+1} =$ 

It is shown in [PR] that the Z^2 action defined as above on the limiting system of this construction is ergodic and weak mixing but not mixing. It is also shown that T and S separately are ergodic.

LEMMA 2.1.1. *For a.e.* $\hat{x} \in [0,1)$, *there exist* N *such that* $\hat{x} \in B_m$, $m \geq N$.

Proof. This follows immediately from the construction. ∎

DEFINITION 2.1.2. If $\hat{x} \in B_n$ then define $\rho = \rho_{\hat{x}}(n) \in \{1, \ldots, 9\}$ to be that integer with $\hat{x} \in B_n^{\rho}$. Thus ρ describes which of the nine slices of B_n $\hat{x}$ lies in.

LEMMA 2.1.3. *For a.e.* $\hat{x} \in [0, 1)$ *and for any block* $b \in \{1, \ldots, 9\}^{k+1}$ *for* k *arbitrary*, $\#\{n : \rho_{\hat{x}}(n) \ldots \rho_{\hat{x}}(n+k) = b\} = \infty$.

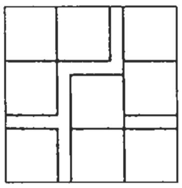
Proof. The values of $\rho_{\hat{x}}(n)$ exactly correspond to the nine-adic expansion of $\hat{x}$ in this interval. Thus the result holds because of the ergodic theorem for $\times 9$ mod 1. ∎

2.2. *The Symbolic Representation*

One can develop a symbolic version of the Z^2-Chacon's that mimics the construction of the last section in the following way. First set $\sigma(0)$ equal to a 3×3 block of zeros,

$$\sigma(0) = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}.$$

Define h_n as before and let $\sigma(n)$ be the $h_n \times h_n$ block of symbols defined by placing the symbols of $\sigma(n-1)$ into each square of the pattern and filling in the rest with 1's:



Define $X \subset \{0, 1\}^{Z^2}$ to be the double infinite sequences x such that each finite block of x is a subblock of some $\sigma(n)$. Let T be the left shift and S the down shift. (X, T, S) is isomorphic to the system described in Section 2.1 by partitioning $[0, 1)$ into $P_1 = [0, 43/47)$ and $P_0 = [43/47, 1)$ and mapping $\hat{x} \in [0, 1)$ to a point $x \in X$ by $x(i, j) = k$ when $T^i S^j(\hat{x}) \in P_k$. The Lebesgue measure on $[0, 1)$ is then mapped to λ on X. The analogs to Lemmas 2.1.1 and 2.1.3 are the following:

LEMMA 2.2.1. *For a.e.* $x \in X$, *there exists* N *such that* $x(0, 0) \in \sigma(m)$ *for* $m \geq N$.

DEFINITION 2.2.2. For $x(0, 0) \in \sigma(n)$ and $x(0, 0) \in \sigma(n+1)$, let $\rho_x(n)$ be the position of $\sigma(n)$ in $\sigma(n+1)$.

LEMMA 2.2.3. *For a.e. $x \in X$ and for any block b in $\{1,\ldots,9\}^{k+1}$ with k arbitrary, $\#\{n : \rho_x(n)\ldots\rho_x(n+k) = b\} = \infty$.*

2.3. *The Time-Zero Partition*

DEFINITION 2.3.1. Let P be the time-zero partition for X, so $P = \{[0],[1]\}$ where $[i] = \{x : x(0,0) = i\}$.

DEFINITION 2.3.2. Given $x \in X$, the $T - P$ name of x is the one dimensional array of symbols $(\ldots x(-1,0), x(0,0), x(1,0),\ldots)$, i.e., it is the horizontal axis of symbols in x.

DEFINITION 2.3.3. P generates under T if $\bigvee_{i=-\infty}^{\infty} T^{-i}P = \mathscr{B}$, the Borel σ-algebra. An equivalent characterization is to say that $\bigvee_{i=-\infty}^{\infty} T^{-i}P$ differentiates points a.e., i.e., there is a set $\tilde{X} \subset X$ of full measure such that if $x, y \in \tilde{X}$ and $x \neq y$ then there exists some N and two different sets $q_1, q_2 \in \bigvee_{i=-N}^{N} T^{-i}P$ such that $x \in q_1$ and $y \in q_2$.

The purpose of this section of the paper is to prove the following theorem, which says that the Z^2 Chacon system has a one dimensional generator.

THEOREM 2.3.4. *P is a generating partition under T.*

Proof. Let $X_o \subset X$ be the set of full measure determined by Lemmas 2.2.1 and 2.2.3. The proof is in two parts. For the first half, recall that each block of $x \in X$ lies in some $\sigma(k)$. Thus there are $\sigma(i)$, $i < k$, inside $\sigma(k)$, some of which intersect the $T - P$ name of x. Mark off where these intersections occur. We will denote such a block of symbols as $\varsigma(i)$, i.e., $\varsigma(i)$ is one of the horizontal lines of symbols from $\sigma(i)$. The following proposition claims this can be done by just looking at the $T - P$ name of x and does not require any two dimensional array of symbols.

PROPOSITION 2.3.5. *Let $x \in X_o$. Then the $T - P$ name of x will determine the locations within it of $\varsigma(i)$ for $i \in N$.*

The proof will be given in Subsection 2.4.

PROPOSITION 2.3.6. *Given the $T - P$ name of a point $x \in X_o$ and the locations within it of $\varsigma(i)$, $i \in N$, we can determine the level of these $\varsigma(i)$ in $\sigma(i)$.*

This proof will be given in Subsection 2.5.

Proof. Continued for Theorem 2.3.4. Take $x, y \in X_o$ distinct points. Thus there exists (i,j) such that $x(i,j) \neq y(i,j)$. If we can find N such that $x(N,0) \neq y(N,0)$ then the cylinder sets from $\bigvee_{i=-N}^{N} T^{-i}P$ given by

$[x(-|N|,0),\ldots,x(|N|,0)]$ and $[y(-|N|,0),\ldots,y(|N|,0)]$ will satisfy the definition. So assume such an N does not exist. Thus x and y have the same $T-P$ name.

By Lemma 2.2.3 we can find m such that $x(0,0)$ and $x(i,j)$ are both in $\sigma(m)$ and likewise for y. By Proposition 2.3.5 we know that $x(0,0)$ and $y(0,0)$ must have the same horizontal position in $\sigma(m)$ and by Proposition 2.3.6 they must have the same vertical position as well. But this means that $x(i,j)=y(i,j)$ and we have reached a contradiction. ▮

2.4. *Horizontal Position*

Recall that $\sigma(n)$ has h_n horizontal lines of symbols, not all of which are distinct. Let $\varsigma(n)$ be an arbitrary line from $\sigma(n)$, fixed within each statement unless otherwise noted.

Notation.

$\varsigma(n)_b$ is the bottom line of $\sigma(n)$.

$\varsigma(n)_t$ is the top line of $\sigma(n)$.

$\varsigma(n)^*$ are the lines in $\sigma(n)$ that could be above the line $\varsigma(n)$.

$\varsigma(n)_*$ are the lines in $\sigma(n)$ that could be below the line $\varsigma(n)$.

1^{h_n} is a block of consecutive 1's of length h_n.

Notice that for $n>0$, $\varsigma(n)_b \neq \varsigma(n)_t$.

LEMMA 2.4.1. *The following is a complete list of predecessors and followers to* $\varsigma(n)$ *that can be found in any* $x \in X$:

Block	*Lines*
$\varsigma(n)\,1\,\varsigma(n)$	$1,\ldots,h_n$
$1^{h_n}\,1\,\varsigma(n)_b$	h_n+1
$\varsigma(n)_b\,1^{h_n}$	h_n+1
$\varsigma(n)\,1\,\varsigma(n)^*$	$h_n+2,\ldots,2h_n$
$\varsigma(n)\,\varsigma(n)_*$	$h_n+2,\ldots,2h_n$
$\varsigma(n)_t\,1\,1^{h_n}$	$2h_n+1$
$1^{h_n}\,1\,\varsigma(n)_t$	$2h_n+1$
$\varsigma(n)\,\varsigma(n)$	$2h_n+2,\ldots,3h_n+1$

Proof. Certainly these all occur, as the line numbers indicate their position in $\sigma(n+1)$. If $\varsigma(n)$ is on the edge of $\sigma(n+1)$, we must look to a larger $\sigma(k)$ to find its predecessors or followers. But the $\sigma(n)$'s on the left and right side of $\sigma(k)$, $n<k$ are at the same vertical positions. Thus placing the $\sigma(k)$'s together as allowed in $\sigma(k+1)$ will yield the same relationships for the $\varsigma(n)$ as listed above. ▮

LEMMA 2.4.2. *Let $k > n$. Then there are at most three (possibly distinct) $\varsigma(n)$'s directly next to each other in $\sigma(k)$.*

Proof. $\sigma(k)$ is constructed by concatenating together many $\sigma(n+1)$'s, with two consecutive $\sigma(n+1)$'s either at the same level or with one shifted vertically by one unit. In the first case, the zigzag of spacers in $\sigma(n+1)$ prevents more than three consecutive $\varsigma(n)$'s. In the second case, the vertical shift causes certain $\varsigma(n)$'s to be placed next to strings of 1's and the zigzag of spacers takes care of the rest. ∎

The proofs of the following lemmas follow directly from the construction of $\sigma(n+1)$. Recall that $\varsigma(n)$ is an arbitrary, fixed line of $\sigma(n)$.

LEMMA 2.4.3. *If $\varsigma(n+1) = \varsigma(n)\,1\,\varsigma(n)\,\varsigma(n)$ then $\varsigma(n+1)_*$ could be*

(i) $\varsigma(n)_*\,1\,\varsigma(n)_*\,\varsigma(n)_*$

(ii) $\varsigma(n)_*\,1\,\varsigma(n)\,\varsigma(n)_*$

(iii) $1^{h_n}\,1\,\varsigma(n)\,1^{h_n}$ *if* $\varsigma(n) = \varsigma(n)_b$.

$\varsigma(n+1)^*$ *could be*

(i) $\varsigma(n)^*\,1\,\varsigma(n)^*\,\varsigma(n)^*$

(ii) $\varsigma(n)\,1\,\varsigma(n)^*\,\varsigma(n)$

(iii) $\varsigma(n)\,1\,1^{h_n}\,\varsigma(n)$ *if* $\varsigma(n) = \varsigma(n)_t$

(iv) $1^{h_n}\,1\,\varsigma(n)_b\,1^{h_n}$ *if* $\varsigma(n) = \varsigma(n)_t$.

LEMMA 2.4.4. *If $\varsigma(n+1) = \varsigma(n)\,\varsigma(n)\,1\,\varsigma(n)$ then $\varsigma(n+1)_*$ could be*

(i) $\varsigma(n)_*\,\varsigma(n)_*\,1\,\varsigma(n)_*$

(ii) $\varsigma(n)_t\,1\,1^{h_n}\,\varsigma(n)_t$ *if* $\varsigma(n) = \varsigma(n)_b$.

$\varsigma(n+1)^*$ *must be* (i) $\varsigma(n)^*\,\varsigma(n)^*\,1\,\varsigma(n)^*$.

LEMMA 2.4.5. *If $\varsigma(n+1) = \varsigma(n)\,1^{h_n}\,1\,\varsigma(n)$ then it must be that $\varsigma(n) = \varsigma(n)_t$ and $\varsigma(n+1)_* = \varsigma(n)_t\,1\,\varsigma(n)_t\,\varsigma(n)_t$ and $\varsigma(n+1)^* = \varsigma(n)_b\,\varsigma(n)_b\,1\,\varsigma(n)_b$.*

LEMMA 2.4.6. *If $\varsigma(n+1) = 1^{h_n}\,1\,\varsigma(n)\,1^{h_n}$ then it must be that $\varsigma(n) = \varsigma(n)_b$ and $\varsigma(n+1)_* = \varsigma(n)_t\,1\,\varsigma(n)_t\,\varsigma(n)_t$ and $\varsigma(n+1)^* = \varsigma(n)_b\,1\,\varsigma(n)_b\,\varsigma(n)_b$.*

PROPOSITION 2.4.7. *Consider the $T-P$ name of $x \in X$. If we know the locations of all $\varsigma(n)$ in this name then we can find the locations of all $\varsigma(n+1)$ in this name.*

Proof. Consider a $\varsigma(n)$ in the $T-P$ name of x. By Lemma 2.4.2 we must have one of

(1) $1\,\varsigma(n)\,1$ (2) $1\,\varsigma_1(n)\,\varsigma_2(n)\,1$ (3) $1\,\varsigma_1(n)\,\varsigma_2(n)\,\varsigma_3(n)\,1$.

[The subscripts are used to indicate that these $\varsigma(n)$'s need not be the same block of symbols.] The theorem is proven by determining, for each case, the location of each $\varsigma(n)$ in $\varsigma(n+1)$, i.e., if it is the left, middle, or right block in $\varsigma(n+1)$. Notice there is no $\varsigma(n+1)$ that does not have at least one $\varsigma(n)$ in it.

The method for finding the location is simple: in each case write all possible ways the case can occur. Then we will show that the $T - P$ name distinguishes these from one another by assuming it doesn't, letting that further dictate the pattern the symbols must take, and reaching a contradiction.

The $\varsigma(n+1)$'s will be indicated by overlined brackets, and the portion of the block that corresponds to the particular case being examined is put in parantheses.

Case 1. *Possible ways to see* $1\,\varsigma(n)\,1$. If $\varsigma(n)$ is the first block in $\varsigma(n+1)$, we could have

(a) $$(1\, \overbrace{\varsigma(n)\, 1)\, \varsigma(n)\, \varsigma(n)}$$

(b) $$(1\, \overbrace{\varsigma(n)\, 1)\, \varsigma(n)^{*}\, \varsigma(n)}$$

(c) $$(1\, \overbrace{\varsigma(n)_t\, 1)\, 1^{h_n}\, \varsigma(n)_t}\,.$$

If $\varsigma(n)$ is the middle block in $\varsigma(n+1)$, the only possibility is

(d) $$\overbrace{1^{h_n}\,(1\, \varsigma(n)_b\, 1)\, 1^{h_n - 1}}\,.$$

If $\varsigma(n)$ is the last block in $\varsigma(n+1)$, we could have

(e) $$\overbrace{\varsigma(n)_t\, 1^{h_n}\,(1\, \varsigma(n)_t}\, 1)$$

(f) $$\overbrace{\varsigma(n)\, \varsigma(n)\, (1\varsigma(n)}\, 1).$$

Lemmas 2.4.5 and 2.4.6 can distinguish (c), (d), and (e) from all other possibilities. When comparing, either the two will immediately look different or there will be no way to extend one to match the other. Below are two of the comparisons; all others are similar.

Proof. Compare (c) *and* (d). For $n > 0$, $\varsigma(n)_t \neq \varsigma(n)_b$, so these are distinguishable.

For $n = 0$, the first step in the inductive process, the $\varsigma(1)$ block in (c), 000 1 111 000, cannot be proceeded by 111 1 because that will not correspond to $\varsigma(1)^*$ or $\varsigma(1)_*$ 1. This $\varsigma(1) \neq \varsigma(1)_t$ or $\varsigma(1)_b$ so those are the only possibilities. ∎

Proof. Compare (e) *and* (a). By Lemma 2.4.1, the only followers for the $\varsigma(n+1)$ in (e) that could start with $1\,\varsigma(n)$ are $1\varsigma(n+1)$ and $1\varsigma(n+1)^*$. By Lemma 2.4.5 we know neither will match the pattern in (a). ∎

This leaves (a), (b), and (f). We do not need to distinguish between (a) and (b) since both are situations where $\varsigma(n)$ is the first block in $\varsigma(n+1)$.

Proof. Compare (a) *and* (f). If they are not distinguishable then it must be that (a) and (f) can be extended to

(a) $$\varsigma(n)\,\varsigma(n)\,(\overbrace{1\,\varsigma(n)\,1)\,\varsigma(n)\,\varsigma(n)}$$

(f) $$\overbrace{\varsigma(n)\,\varsigma(n)\,(1\,\varsigma(n)}\,1)\,\varsigma(n)\,\varsigma(n).$$

By Lemma 2.4.1 the only way the $\varsigma(n+1)$ in (a) can be proceeded so as to match (f) is by $\varsigma(n+1)\,1$ or $\varsigma(n+1)_*\,1$. Using Lemma 2.4.3 the only possibility is

(a) $$\overbrace{\varsigma(n)\,1\,\varsigma(n)\,\varsigma(n)}\,(\overbrace{1\,\varsigma(n)\,1)\,\varsigma(n)\,\varsigma(n)}.$$

Similarly, the only way to extend (f) consistent with (a) is

(f) $$\overbrace{\varsigma(n)\,\varsigma(n)\,(1\,\varsigma(n)}\,1)\,\overbrace{\varsigma(n)\,\varsigma(n)\,1\,\varsigma(n)}.$$

So (a) must be followed by $1\,\varsigma(n)$. The second $\varsigma(n+1)$ in (a) can be followed by $1\,\varsigma(n+1)$ or $1\,\varsigma(n+1)^*$. In either case, (a) extends to

(a) $$\overbrace{\varsigma(n)\,1\,\varsigma(n)\,\varsigma(n)}\,(1\,\overbrace{\varsigma(n)\,1)\,\varsigma(n)\,\varsigma(n)}\,1\,\overbrace{\varsigma(n)\,1\ldots}$$

which in turn implies that (f) must extend to the form

(f) $$1\,\varsigma(n+1)\,1\,\varsigma(n+1)\,1.$$

But for such a $\varsigma(n+1)$, this form does not exist in any $x \in X$. This can be seen by attempting to determine how these $\varsigma(n+1)$ can fit into $\varsigma(n+2)$ and finding that they cannot. ∎

Proof. Compare (b) *and* (f). From Lemma 2.4.1 we see that the $\varsigma(n+1)$ in (f) can only be followed by $1\,\varsigma(n+1)$ or $1\,\varsigma(n+1)^*$. If it is the former we must have $\varsigma(n)^* = \varsigma(n)$. If it is the latter, use Lemma 2.4.4 to conclude the same thing. Thus it reduces to comparing (a) and (f). ∎

Case 2. *Possible ways to see* $1\,\varsigma_1(n)\,\varsigma_2(n)\,1$. Their positions in $\varsigma(n+1)$ could be

(i) $\overbrace{\dots 1\,\varsigma_1(n)}\ \overbrace{\varsigma_2(n)\,1\dots}$

(ii) $1\,\overbrace{\varsigma_1(n)\,\varsigma_2(n)\,1\dots}$

(iii) $\overbrace{\dots 1\,\varsigma_1(n)\,\varsigma_2(n)}\,1.$

If $\varsigma_1(n) \neq \varsigma_2(n)$, these can be extended to

(i) $\overbrace{\varsigma(n)_b\,\varsigma(n)_b\,(1\,\varsigma(n)_b}\ \overbrace{\varsigma(n)_t\,1)\,1^{h_n}\,\varsigma(n)_t}$

(ii) this cannot be extended in any valid way

(iii) $\overbrace{\varsigma(n)\,(1\,\varsigma(n)^*\varsigma(n)}\,1)\,\overbrace{\varsigma(n)\,1\,\varsigma(n)^*\varsigma(n)}$

$\overbrace{\varsigma(n)\,(1\,\varsigma(n)^*\varsigma(n)}\,1)\,\overbrace{\varsigma(n)^*\,1\,\varsigma(n)^{**}\varsigma(n)^*}.$

Then (i) can be distinguished from (iii) because $1^{h_n} \neq \varsigma(n)$ or $\varsigma(n)^*$.

If $\varsigma_1(n) = \varsigma_2(n)$ then there are more valid ways to extend these.

(i) $\overbrace{\varsigma(n)\,1^{h_n}\,(1\,\varsigma(n)}\ \overbrace{\varsigma(n)\,1)\,1^{h_n}\,\varsigma(n)_t}$

$\overbrace{\varsigma(n)_t\,1^{h_n}\,(1\,\varsigma(n)_t}\ \overbrace{\varsigma(n)_t\,1)\,\varsigma(n)_t\,\varsigma(n)_t}$

(ii) $\overbrace{\varsigma(n)\,\varsigma(n)\,1\,\varsigma(n)}\,(1\,\overbrace{\varsigma(n)\,\varsigma(n)\,1)\,\varsigma(n)}$

$\overbrace{\varsigma(n)_*\,\varsigma(n)_*\,1\,\varsigma(n)_*}\,(1\,\overbrace{\varsigma(n)\,\varsigma(n)\,1)\,\varsigma(n)}$

$\overbrace{\varsigma(n)_t\,1\,1^{h_n}\,\varsigma(n)_t}\,(1\,\overbrace{\varsigma(n)_b\,\varsigma(n)_b\,1)\,\varsigma(n)_b}$

$1^{h_{n+1}}\,(1\,\overbrace{\varsigma(n)_b\,\varsigma(n)_b\,1)\,\varsigma(n)_b}$

(iii) $\overbrace{\varsigma(n)\,(1\,\varsigma(n)\,\varsigma(n)\,1)}$ with various followers.

Everything in (i) can be distinguished from (ii) and (iii) because (i) has the form: (not 1's) 1^{h_n} proceeding $1\,\varsigma(n)\,\varsigma(n)\,1$ and that does not appear elsewhere. In (ii) the first three can be easily distinguished because $1\,\varsigma(n)\,\varsigma(n)\,1$ is proceeded by $1\,\varsigma(n)\,1$ and we can use Case 1 to determine the location of that block and thus the two blocks following it. The last one in (ii) can be distinguished from (iii) because it is proceeded by 1's. ∎

Case 3. *Possible ways to see* $1\,\varsigma_1(n)\,\varsigma_2(n)\,\varsigma_3(n)\,1$. These could be grouped as

(i) $$\overbrace{\ldots 1\varsigma_1(n)}\ \overbrace{\varsigma_2(n)\,\varsigma_3(n)\,1\ldots}, \text{ or}$$

(ii) $$\overbrace{\ldots 1\varsigma_1(n)\,\varsigma_2(n)}\ \overbrace{\varsigma_3(n)1\ldots}.$$

If $\varsigma_2(n) \neq \varsigma_3(n)$ then it must be (ii) since the second $\varsigma(n+1)$ in (i) does not permit two different blocks. So assume that $\varsigma_2(n) = \varsigma_3(n)$. If $\varsigma_1(n)$ is different than the other two, then (i) must be extended as

(i) $$\overbrace{\varsigma(n)^*\,\varsigma(n)^*\,(1\,\varsigma(n)^*}\ \overbrace{\varsigma(n)\,\varsigma(n)\,1)\,\varsigma(n)}.$$

The first $\varsigma(n+1)$ in (ii) must be of the form $\varsigma(n)\,1\,\varsigma(n)^*\,\varsigma(n)$, yet comparing (i) and (ii) yields $\varsigma(n) = \varsigma(n)^* = \varsigma_1(n)$, contrary to our assumption. Thus it must be that all the $\varsigma_i(n)$ are the same block.

Notice both (i) and (ii) have the $\varsigma(n+1)$'s directly next to each other which means we must see $\varsigma(n+1)\,\varsigma(n+1)$ or $\varsigma(n+1)\,\varsigma(n+1)_*$. In (i), let $\varsigma(n+1) = \varsigma(n)\varsigma(n)1\varsigma(n)$ and by Lemma 2.4.4 the only consistent $\varsigma(n+1)^*$ is $\varsigma(n)^*\varsigma(n)^*1\varsigma(n)^*$. To match (ii) we must have $\varsigma(n)^* = \varsigma(n)$. Similarly, in (ii) we must have $\varsigma(n+1) = \varsigma(n)1\varsigma(n)\varsigma(n)$ and the only follower that matches (i) is the same $\varsigma(n+1)$. Thus we have

(i) $$\overbrace{\varsigma(n)\,\varsigma(n)\,(1\,\varsigma(n)}\ \overbrace{\varsigma(n)\,\varsigma(n)\,1)\,\varsigma(n)}$$

(ii) $$\overbrace{\varsigma(n)\,(1\,\varsigma(n)\,\varsigma(n)}\ \overbrace{\varsigma(n)\,1)\,\varsigma(n)\,\varsigma(n)}.$$

Thus (i) must be followed by $\varsigma(n)$ and (ii) proceeded by $\varsigma(n)$. As before we find that the extensions are

(i)
$$\overbrace{\varsigma(n)\,\varsigma(n)\,(1\,\varsigma(n)}\ \overbrace{\varsigma(n)\,\varsigma(n)\,1)\,\varsigma(n)}\ \overbrace{\varsigma(n)\,\varsigma(n)\,1\,\varsigma(n)}$$

(ii) $$\overbrace{\varsigma(n)\,1\,\varsigma(n)\,\varsigma(n)}\ \overbrace{\varsigma(n)\,(1\,\varsigma(n)\,\varsigma(n)}\ \overbrace{\varsigma(n)\,1)\,\varsigma(n)\,\varsigma(n)}.$$

But now the only way to match (ii) to (i) is to follow with another $\varsigma(n+1)$, which gives four $\varsigma(n+1)$ in a row. This contradicts Lemma 2.4.2. ∎

Proof of Proposition 2.3.4. For $x \in X_o$ there will be infinitely many 0's and 1's in its $T - P$ name. This lets us start the first step of the induction with the observation that zeros appear only in groups of three so we can easily determine locations of $\varsigma(0)$. For the induction step, assume we know the locations of all $\varsigma(n)$. For each $\varsigma(n)$, determine if it is in Case 1, 2, or 3.

Then consider the $2h_{n+1}$ symbols to each side of $\varsigma(n)$ and how the other $\varsigma(n)$'s are placed there; determine a $\varsigma(n+1)$ pattern consistent for this block. By Proposition 2.4.1 this is the only way $\varsigma(n)$ can be fitted into a $\varsigma(n+1)$ and we now know the location for this $\varsigma(n+1)$. ∎

2.5. *Vertical Position*

DEFINITION 2.5.1. A spacer line is any $\varsigma(n)$ in $\sigma(n)$ with at least three consecutive 1's.

LEMMA 2.5.2. *Spacer lines are unique.*

Proof. We know that to be true in $\sigma(1)$. Assume it is true for $\sigma(n)$. Let I be the collection of spacer lines is $\sigma(n)$. Then the collection of spacer lines for $\sigma(n+1)$ is $\{i, i+h_n, i+h_n+1, i+2h_n+1, h_n+1, 2h_n+1 : i \in I\}$.

Notice that the lines at $h_n + 1$ and $2h_n + 1$ are the only lines in $\sigma(n)$ with h_n consecutive 1's and thus are certainly unique.

The lines $i, i+h_n+1$, and $i+2h_n+1$ all have the same first h_n symbols, $\varsigma(n)$, and are the only lines in $\sigma(n+1)$ to do so, by induction hypothesis. They are followed by $1\,\varsigma(n)\,\varsigma(n)$, $1\,\varsigma(n)^*\,\varsigma(n)$, and $\varsigma(n)\,1\,\varsigma(n)$, respectively. The first two are not equal by the induction hypothesis and the first and last are not equal because $\varsigma(n) \neq 1^{h_n}$. If the last two are equal it must be that $\varsigma(n)$ starts with a $\varsigma(k+1)$, $k < n$, of the form $1^{h_n}\,1\,\varsigma(k)_b\,1^{h_k}$ and thus $\varsigma(n)^*$ starts with $1^{h_k}\,\varsigma(k)_b$ which is inconsistent with $\sigma(k+1)$.

The spacer lines $i + h_n$ have the form (symbols from line $i-1$) 1 (line i) (line $i-1$) and the induction hypothesis again ensures they are unique. ∎

LEMMA 2.5.3. *$\sigma(n)$ can never have more than three consecutive non-spacer lines.*

Proof. This is clearly true in $\sigma(1)$. Assume it is true for $\sigma(n)$. The $\sigma(n)$'s in $\sigma(n+1)$ are always separated vertically by a spacer line, so the result holds for $\sigma(n+1)$. ∎

For the following, let N be large enough so that $x(0,0) \in \sigma(m)$ for $m \geq N$. We know such an N exists by Lemma 2.2.1.

LEMMA 2.5.4. *Let $m \geq N$ be fixed and let W represent lines from $\sigma(m)$. Assume the $T - P$ name of x around $x(0,0)$ has the pattern*

$$\overbrace{W1WW\,1\,W1WWWW1W}\,1\,\overbrace{W1WW\,1\,W1\dot{W}WWW1W}\,\overbrace{W1WW\,1\,W1WWWW1W},$$

where the overline brackets indicate $\varsigma(m+2)$'s and $\dot{W}$ represents the particular $\varsigma(m)$ that $x(0,0)$ lies in. If the symbols are consistent with $\rho_x(m)\,\rho_x(m+$

$1)\rho_x(m+2) = 555$ then it must be the case that indeed $\rho_x(m)\rho_x(m+1)\rho_x(m+2) = 555$.

Proof. The pattern written above is consistent not only with 555 but also with 222, 225, 252, 255, 522, 525, and 552. Let α be the particular $\varsigma(m)$ that $x(0,0)$ lies in, so $\dot{W} = \alpha$. Since we are assuming that the pattern is consistent with 555 we can write the pattern, starting with the $\varsigma(m)$ that $x(0,0)$ lies in, as

$$\overbrace{\alpha\alpha_* \ \alpha_{**}1\alpha_*\alpha_{**}} \ \overbrace{\alpha_{***}1\alpha_{**}\alpha_{***}\ 1\ \alpha_{**}1\alpha_*\alpha_{**}\ \alpha_{***}1\alpha_{**}\alpha_{***}}.$$

If the symbols are also consistent with another number, then we must be able to also write the pattern as one of

$$
\begin{array}{lll}
222 & \overbrace{\alpha\alpha\ \alpha1\alpha\alpha} & \overbrace{\alpha1\alpha\alpha\ 1\ \alpha1\alpha\alpha\ \alpha1\alpha\alpha} \\
225 & \overbrace{\alpha\alpha\ \alpha1\alpha\alpha} & \overbrace{\alpha_*1\alpha_*\alpha_*\ 1\ \alpha_*1\alpha_*\alpha_*\ \alpha_*1\alpha_*\alpha_*} \\
252 & \overbrace{\alpha\alpha\ \alpha_*1\alpha_*\alpha_*} & \overbrace{\alpha_*1\alpha_*\alpha_*\ 1\ \alpha1\alpha\alpha\ \alpha_*1\alpha_*\alpha_*} \\
255 & \overbrace{\alpha\alpha\ \alpha_*1\alpha_*\alpha_*} & \overbrace{\alpha_{**}1\alpha_{**}\alpha_{**}\ 1\ \alpha_*1\alpha_*\alpha_*\ \alpha_{**}1\alpha_{**}\alpha_{**}} \\
522 & \overbrace{\alpha\alpha_*\ \alpha_*1\alpha\alpha_*} & \overbrace{\alpha_*1\alpha\alpha_*\ 1\ \alpha_*1\alpha\alpha_*\ \alpha_*1\alpha\alpha_*} \\
525 & \overbrace{\alpha\alpha_*\ \alpha_*1\alpha\alpha_*} & \overbrace{\alpha_{**}1\alpha_*\alpha_{**}\ 1\ \alpha_{**}1\alpha_*\alpha_{**}\ \alpha_{**}1\alpha_*\alpha_{**}} \\
552 & \overbrace{\alpha\alpha_*\ \alpha_{**}1\alpha_*\alpha_{**}} & \overbrace{\alpha_{**}1\alpha_*\alpha_{**}\ 1\ \alpha_*1\alpha\alpha_*\ \alpha_{**}1\alpha_*\alpha_{**}}.
\end{array}
$$

But in each case, if the symbols are consistent with 555 and another scenario then it must be that $\alpha = \alpha_* = \alpha_{**} = \alpha_{***}$. Yet there is no line in $\sigma(n)$ which is the same as the three lines underneath it.

Proof of Proposition 2.3.6. Use Lemma 2.2.1 to take N so large that $x(0,0) \in \sigma(m)$ for $m \geq N$. By Lemmas 2.2.3 and 2.5.4 we can find $M \geq \max\{N, i\}$ so that $\rho_x(M)\rho_x(M+1)\rho_x(M+2) = 555$. If $\alpha = \varsigma(M)$ is the block $x(0,0)$ lies in, then the $T-P$ name of x can be written in part as $\alpha\alpha_*\alpha_{**}1\alpha_*\alpha_{**}\ \alpha_{***}\ldots$. By Lemma 2.5.3 one of these will be a spacer line. If $x(k,0)$ is in this spacer line, by Lemma 2.5.2 we will know exactly where $x(k,0)$ lies vertically in $\sigma(M)$. But then we can move back to $x(0,0)$ and find its vertical position, which tells us the vertical position of $x(0,0)$ in $\sigma(i)$. ∎

3. THE CENTRALIZER OF T

DEFINITION 3.1. $C(T)$ is the set of (not necessarily invertible) measure-preserving transformations ϕ on (X, λ) with $\phi T = T\phi$. Note that we do not assume that ϕ commutes with S, the down shift.

The main purpose of this paper is to explore $C(T)$, and the resulting theorem is as follows:

THEOREM 3.2. $C(T) = \{T^i S^j\}_{(i,j) \in Z^2}$.

Certainly $\{T^i S^j\}_{(i,j) \in Z^2} \subset C(T)$. What needs to be shown is that every $\phi \in C(T)$ has a pair (i, j) such that $\phi = T^i S^j$ a.e.

Because of the result of Section 2, which says that each $x \in X_o$ can be written as a 1-dimensional array of symbols, we can prove Theorem 3.2 in a manner similar to that used by A. delJunco [dJ] for the 1-dimensional Chacon transformation. The main difference is that now blocks in x and $\phi(x)$ can take on many forms and we must consider these different cases.

3.1. *Distinguishing Points and Blocks*

Recall from the symbolic construction of X that for a.e. x there is an N such that, for $n \geq N$, $x(0,0)$ lies in a symbolic block $\sigma(n)$ of size $h_n \times h_n$ and this $\sigma(n)$ has position k, $k \in \{1, \dots, 9\}$, in $\sigma(n+1)$. The following lemma from [PR] relates these positions for two points.

LEMMA 3.1.1. *There exists $D \subset X$ with full measure so that for all $x, y \in D$ if $y \neq T^i S^j x$ for any $(i, j) \in Z^2$ then there are infinitely many positive integers n such that*

(i) *the time-zero coordinates of x and y lie in different n-blocks of their $(n+1)$-blocks, and*

(ii) *the n-blocks containing the time-zero coordinates of the names x and y overlap in a rectangle with sides of length at least $h_n/10$.*

We can rewrite (i) by letting $(k, l)_n$ be the pair in $\{1, \dots, 9\}^2$ such that k describes the position of $\sigma(n)$ in $\sigma(n+1)$ for x, as above, and l does the same for $\phi(x)$. Then the above lemma says that if $\phi(x) \neq T^i S^j x$ then there are infinitely many n such that $(k, l)_n \notin \{(i, i) : i = 1, \dots, 9\}$ and such that their n-blocks have sufficiently large overlap. We will prove Theorem 3.2 by showing such an infinite set of n does not exist. We will continue to make use of Proposition 2.3.5 which says we can determine where the blocks $\sigma(n)$ intersect the horizontal array of symbols. In particular, let $\varsigma_x(n)$ be the intersection which includes $x(0,0)$. Depending on the location of $\sigma(n)$ in $\sigma(n+1)$, the $\varsigma_x(n+1)$ block will be of the form

(i) $\varsigma_x(n)\,1\,\varsigma_x(n)\,\varsigma_x(n)$

(ii) $\varsigma_x(n)\,1\,\varsigma_x(n)^*\,\varsigma_x(n)$ or $\varsigma_x(n)_*\,1\,\varsigma_x(n)\,\varsigma_x(n)_*$

(iii) $\varsigma_x(n)\,\varsigma_x(n)\,1\,\varsigma_x(n)$.

We need to be able to distinguish these cases, thus we need x such that $\varsigma_x(n) \neq \varsigma_x(n)^*$ and similarly, $\varsigma_x(n) \neq \varsigma_x(n)_*$.

LEMMA 3.1.2. *Let $E_n = \{x \mid \varsigma_x(n) = \varsigma_x(n)_*\}$. These are the points which lie on a line of $\sigma(n)$ which is the same as the line below it. Let $F_n = \{x \mid \varsigma_x(n) = \varsigma_x(n)^*\}$. These are the points which lie on a line of $\sigma(n)$ which is the same as the line above it. Then*

$$\lambda\Big(\bigcap_k \bigcup_{n\geq k} E_n\Big) = 0 \quad \textit{and} \quad \lambda\Big(\bigcap_k \bigcup_{n\geq k} F_n\Big) = 0.$$

Hence for a.e. x there exists an $N = N(x)$ such that $x \in E_n^c$ and $x \in F_n^c$ for every $n \geq N$.

Proof. We will prove the first statement. The second is done similarly. Notice that $E_{k+1} \subset E_k$, so $\bigcup_{n\geq k} E_n = E_k$ and $\lambda(\bigcap_k \bigcup_{n\geq k} E_n) = \lim_{k\to\infty} \lambda(E_k)$. Thus we just need to estimate $\lambda(E_k)$. This will be done by induction, using the following:

(i) Recall that B_1 contains 100 subintervals of length l_1, arranged into 10 horizontal rows. Each row altogether has measure less than $\frac{1}{10}$.

(ii) There are 2^k rows in $\sigma(k)$ which are the same as both the rows above and below them. This is clear for $k = 1$, and if assumed at step n then $n+1$ follows because $\sigma(n+1)$ is constructed from $\sigma(n)$ in such a way that the top and bottom third will each have 2^n such rows. The shift in the middle third will prevent it from occurring there.

(iii) If $\varsigma(n)$ is a row such that $\varsigma(n) = \varsigma(n)^* = \varsigma(n)_*$, then and only then will the associated $(n+1)$-block $\varsigma(n+1) = \varsigma(n)1\varsigma(n)^*\varsigma(n)$ be equal to $\varsigma(n+1)_*$.

Now we can do the induction.

First, notice by inspection that 5 out of the 10 rows of $\sigma(1)$ are the same as the row below it. This corresponds to the points in those 5 rows of B_1 so by (i), $\lambda(E_1) < 5 \times \frac{1}{10}$.

E_2 is that subset of E_1 which includes the front and back third of each row of B_1 included in E_1 but by (iii) only includes the middle third for those rows which corresponded in $\sigma(1)$ to rows equaling both the rows above and below them. By (ii) there are only 2 such rows and we have $\lambda(E_2) < \frac{2}{3} \times \frac{5}{10} + \frac{1}{3} \times 2 \times \frac{1}{10}$. Now assume $\lambda(E_n) < (\frac{2}{3})^{n-1} \times \frac{5}{10} + \frac{1}{3} \times \frac{2}{10} \times (n-1) \times (\frac{2}{3})^{n-2}$. Then E_{n+1} will be that subset of E_n which includes the front and back third of each row of B_n included in E_n but only includes the middle third for those rows which correspond in $\sigma(n)$ to rows equaling both the rows above and below them. By (ii) there are 2^n such rows and we have $\lambda(E_{n+1}) < \frac{2}{3}[(\frac{2}{3})^{n-1} \times \frac{5}{10} + \frac{1}{3} \times \frac{2}{10} \times (n-1) \times (\frac{2}{3})^{n-2}] + 2^n \times \frac{1}{10} \times (\frac{1}{3})^n = (\frac{2}{3})^n \times \frac{5}{10} + \frac{1}{3} \times \frac{2}{10} \times n \times (\frac{2}{3})^{n-1}$. Now it is clear that as $n \to \infty$, $\lambda(E_n) \to 0$ and we have the result. ∎

Let $E = (\bigcap_k \bigcup_{n \geq k} E_n)^c$ and $F = (\bigcap_k \bigcup_{n \geq k} F_n)^c$. Then Lemma 3.1.2 shows $\lambda(E) = 1 = \lambda(F)$, which says that there is, for a.e. x, some block size N such that for $n \geq N$, $\varsigma_x(n)$ is distinguishable from $\varsigma_x(n)^*$ and $\varsigma_x(n)_*$.

When comparing blocks, we will need that not only are the n-blocks distinguishable but also the $(n-4)$-subblocks within the n-block. Let $G_n = \{x \mid \text{all } (n-4)\text{-subblocks of } \varsigma_x(n) \text{ are different than the corresponding } (n-4)\text{-subblocks of } \varsigma_x(n)^* \text{ and } \varsigma_x(n)_*\}$. Let $\bar{F}_n^c$ be a subset of F_n^c defined as

$$\begin{aligned}\bar{F}_n^c = \{x : &\varsigma_x(n) \neq \varsigma_x(n)_*, \varsigma_x(n)_* \neq \varsigma_x(n)_{**}, \varsigma_x(n)_{**} \\ &\neq \varsigma_x(n)_{***}, \varsigma_x(n)_{***} \neq \varsigma_x(n)_{****}, \varsigma_x(n)_{****} \\ &\neq \varsigma_x(n)_{*****}, \varsigma_x(n) \neq \varsigma_x(n)^*, \varsigma_x(n)^* \\ &\neq \varsigma_x(n)^{**}, \varsigma_x(n)^{**} \neq \varsigma_x(n)^{***}, \varsigma_x(n)^{***} \\ &\neq \varsigma_x(n)^{****}, \varsigma_x(n)^{****} \neq \varsigma_x(n)^{*****}\}.\end{aligned}$$

LEMMA 3.1.3. $\mu(\bar{F}_n^c) \geq \mu(F_n^c) - 10\mu(F_n)$.

Proof. Consider a row ς in B_n such that the points x on this row are in F_n^c. If ς and the four rows above and below ς are each different from the row above and below it, then the points in ς are in $\bar{F}_n^c$. Each row of B_n with points in F_n eliminates at most 10 rows of points from F_n^c, hence the result. ∎

Thus we have that $\mu(\bar{F}_n^c) \to 1$ as $n \to \infty$. The following lemma is clear:

LEMMA 3.1.4. $\bar{F}_n^c \subset \bar{F}_{n+1}^c$

Thus we have for a.e. x, there exists N such that for every $n \geq N$, $x \in \bar{F}_n^c$. But $\bar{F}_n^c \subset G_{n+4}$, so we can say there is a set G of full measure for which the "good subblock" condition holds, i.e., for $x \in G$, there exists an M such that for every $m \geq M$, all $(m-4)$-subblocks of $\varsigma_x(m)$ are different than the corresponding $(m-4)$-subblocks of $\varsigma_x(m)^*$ and $\varsigma_x(m)_*$.

Let $\tilde{X} = X_o \cap D \cap E \cap F \cap G$. Then $\lambda(\tilde{X}) = 1$ and for the remainder of this paper we will take points x such that x and $\phi(x)$ are in $\tilde{X}$.

3.2. *Finite Codes and d-Distances*

Let α be a block of symbols of length $|\alpha|$, and β another string of symbols of the same length. Define $d(\alpha, \beta) = (1/|\alpha|)\#\{i \mid \alpha(i) \neq \beta(i)\}$. We generalize this to strings of symbols $x, y \in \{0,1\}^Z$ by $d(x, y) = \lim_{i,j\to\infty} d(x[-i, j], y[-i, j])$, if the limit exists, where $x[-i, j]$ is the block of symbols starting at $x(-i)$ and ending at $x(j)$. We say α occurs at position i in x if $x[i, i + |\alpha| - 1] = \alpha$.

A finite code is a map φ for which there exists an N such that $\varphi(x)(0)$ depends only on $x[-n, n]$. We then write $|\varphi| < n$. Since points in (X, λ) are determined by a 1-dimensional string of symbols, every transformation ϕ of (X, λ) which commutes with the shift T can be approximated by a finite code φ. Thus for every ϵ there exists a finite code φ such that $d(\phi(x), \varphi(x)) < \epsilon$ for a.e. x.

With these two concepts we can state the following lemmas.

LEMMA 3.2.1. *Suppose α occurs at position i in $\varsigma(n)$ $(n \geq 5)$, $|\alpha| > h_{n-3}$, and β is the string of symbols of the same length that occur at position i in $\varsigma(n)^*$, where we assume $\varsigma(n) \neq \varsigma(n)^*$ and each $(n-4)$-subblock of $\varsigma(n)$ does not equal the associated $(n-4)$-subblock of $\varsigma(n)^*$. Then $d(\alpha, \beta) > 10^{-4}$.*

Proof. First we show that $d(\varsigma(n), \varsigma(n)^*) > 10^{-4}$. Recall that $\sigma(k)$ is constructed using nine copies of $\sigma(k-1)$ and inserting 1's as "spacers". For most lines $\varsigma(k)$ and $\varsigma(k)^*$, there is one common position in which a spacer occurs. This can happen at each step in the construction and can result in the two lines having many "agreeing spacers". Let I_n be the set of i indicating these positions in $\varsigma(n)$. Then $|I_n| \leq \frac{1}{2}(3^n - 1)$. Since $h_n = \frac{1}{2}(3^n 7 - 1)$, for large enough n these spacers will take up at most $\frac{1}{3}$ of the symbols in a $\varsigma(n)$.

Now let us focus on the rest of the indices. Let $\delta(\varsigma(k), \varsigma(k)^*)$ be the proportion of indices such that $\varsigma(k)(i) \neq \varsigma(k)^*(i)$ and $i \notin I_k$. Then $\delta(\varsigma(1), \varsigma(1)^*)$, as $\varsigma(1)$ ranges consecutively from the bottom row of $\sigma(1)$ to the next to top row, is

[1] $$0, 0, \tfrac{6}{9}, \tfrac{6}{9}, 0, \tfrac{3}{9}, \tfrac{3}{9}, 0, 0.$$

One can think of the rows in $\sigma(2)$ as having the form

(i) $\varsigma(1)\,1\,\varsigma(1)\,\varsigma(1)$ or $\varsigma(1)\,\varsigma(1)\,1\,\varsigma(1)$, in which case $\delta(\varsigma(2), \varsigma(2)^*)$ is the same as $\delta(\varsigma(1), \varsigma(1)^*)$.

(ii) $1^{h_1}\,1\,\varsigma(1)\,1^{h_1}$ or $\varsigma(1)\,1\,1^{h_1}\,\varsigma(1)$, in which case the δ distance is trivially $\geq \frac{2}{9}$.

(iii) $\varsigma(1)\,1\,\varsigma(1)^*\,\varsigma(1)$, in which case $\delta(\varsigma(2), \varsigma(2)^*) = \frac{2}{3}\delta(\varsigma(1), \varsigma(1)^*) + \frac{1}{3}\delta(\varsigma(1)^*, \varsigma(1)^{**})$. Using [1], we see that this yields, consecutively, the δ-distances of

[2] $$0, \tfrac{6}{27}, \tfrac{18}{27}, \tfrac{12}{27}, \tfrac{3}{27}, \tfrac{9}{27}, \tfrac{6}{27}, 0, \tfrac{9}{27}.$$

Repeat these three steps for $\sigma(3)$:

(i) yields the same δ-distances found in [1],

(ii) yields at least $\frac{2}{9}$,

(iii) yields the δ-distances found both in [2] and, using [2] with above formula,

[3] $$\tfrac{6}{81}, \tfrac{30}{81}, \tfrac{48}{81}, \tfrac{27}{81}, \tfrac{15}{81}, \tfrac{24}{81}, \tfrac{12}{81}, \tfrac{9}{81}, \tfrac{31}{81}.$$

Notice that $\delta(\varsigma(3), \varsigma(3)^*) \geq \frac{2}{27}$ if it is not 0. We can now finish the proof by induction. Assume $\delta(\varsigma(n), \varsigma(n)^*) \geq \frac{2}{27}$ if it is not 0. Then the lines of $\sigma(n+1)$ can come in one of three types:

(i) $\varsigma(n)\,1\,\varsigma(n)\,\varsigma(n)$ or $\varsigma(n)\,\varsigma(n)\,1\,\varsigma(n)$, in which case $\delta(\varsigma(n+1), \varsigma(n+1)^*) = \delta(\varsigma(n), \varsigma(n)^*) \geq \frac{2}{27}$ if it is not 0.

(ii) $1^{h_n}\,1\,\varsigma(n)\,1^{h_n}$ or $\varsigma(n)\,1\,1^{h_n}\varsigma(n)$, in which case $\delta(\varsigma(n+1), \varsigma(n+1)^*) \geq \frac{2}{27}$ trivially.

(iii) $\varsigma(n)\,1\,\varsigma(n)^*\,\varsigma(n)$, in which case $\delta(\varsigma(n+1), \varsigma(n+1)^*) = \frac{2}{3}\delta(\varsigma(n), \varsigma(n)^*) + \frac{1}{3}\delta(\varsigma(n)^*, \varsigma(n)^{**})$. But $\delta(\varsigma(n), \varsigma(n)^*)$ and $\delta(\varsigma(n)^*, \varsigma(n)^{**})$ must be listed consecutively in one of the lists [1], [2], or [3], thus the combination yields $\sigma(\varsigma(n+1), \varsigma(n+1)^*) \geq \frac{2}{27}$ if not 0.

This completes the induction.

Then $d(\varsigma(n), \varsigma(n)^*) \geq \frac{2}{3}\delta(\varsigma(n), \varsigma(n)^*) \geq \frac{2}{3} \times \frac{2}{27}$ if not zero.

Now let α, β be strings of symbols occurring in $\varsigma(n), \varsigma(n)^*$, respectively. Since $|\alpha| > h_{n-3}$, there is a $\varsigma(n-4), \varsigma(n-4)^*$ occurring at the same position in α and β. Thus

$$d(\alpha, \beta) = \frac{\#\{i : \varsigma(n-4)(i) \neq \varsigma(n-4)^*(i)\}}{|\alpha|} > \frac{4}{81}\,\frac{h_{n-4}}{|\alpha|} > 10^{-4}. \quad \blacksquare$$

LEMMA 3.2.2. *Suppose α occurs at position i in $\varsigma(n)$ $(n \geq 5)$, $|\alpha| > h_{n-3}$, and β is the string of symbols of the same length at position $i+1$. Then $d(\alpha, \beta) > 10^{-4}$.*

Proof. Let $\overline{\varsigma(n)}, \underline{\varsigma(n)}$ be $\varsigma(n)$ with the first (last) symbol removed. Notice the following:

(1) $d(\overline{\varsigma(1)}, \underline{\varsigma(1)}) = \frac{2}{10}$ for every $\varsigma(1)$.

(2) If $\varsigma(k)$ is of the form

$$1^{h_{k-1}}\,1\,\varsigma(k-1)\,1^{h_{k-1}} \quad \text{or} \quad \varsigma(k-1)\,1\,1^{h_{k-1}}\,\varsigma(k-1),$$

then $d(\overline{\varsigma(k)}, \underline{\varsigma(k)}) \geq \frac{1}{28}$ for every k. This follows because the $\varsigma(k-1)$ in the above forms is such that $d(\overline{\varsigma(k-1)}, \underline{\varsigma(k-1)}) \geq \frac{1}{7}$ and it takes up at least $\frac{1}{4}$ of the string of symbols.

(3) As mentioned in the proof on Lemma 3.2.1, the set of indices corresponding to "agreeing spacers" takes up at most $\frac{1}{3}$ of the symbols in $\varsigma(n)$.

$\varsigma(n)$ consist of subblocks of types 1 and 2 and the "agreeing spacers" mentioned in (3). Since $\frac{2}{3}$ of $\varsigma(n)$ is NOT these spacers, we know that $d(\overline{\varsigma(n)}, \underline{\varsigma(n)}) \geq \frac{2}{3} \times \frac{1}{28}$. Now let α, β be as described in the statement of the lemma. Since $|\alpha| > h_{n-3}$, there must be a $\overline{\varsigma(n-4)}$ and $\underline{\varsigma(n-4)}$ occurring in the same position in α, β, respectively. Thus

$$d(\alpha, \beta) > \frac{\#\{i : \varsigma(n-4)(i) \neq \varsigma(n-4)(i-1)\}}{|\alpha|}$$

$$> \frac{2}{3}\,\frac{1}{28}\,\frac{h_{n-4}}{h_n} > 10^{-4}. \quad \blacksquare$$

LEMMA 3.2.3. *Suppose α occurs at position i in $\varsigma(n)$ $(n \geq 5)$, $|\alpha| > h_{n-3}$, and β is the string of symbols of the same length that occurs at position $i+1$ in $\varsigma(n)^*$, $\varsigma(n)_*$, or $\varsigma(n)^{**}$. Then $d(\alpha, \beta) > 10^{-4}$.*

Proof. Let $\overline{\varsigma(n)}$ be $\varsigma(n)$ with the first symbol removed and $\underline{\varsigma(n)_*}$ be $\varsigma(n)_*$ with the last symbol removed. Then this lemma can be proven in the same way as Lemma 3.2.2, with the following changes:

(1) $d(\overline{\varsigma(1)}, \underline{\varsigma(1)_*}) \geq \frac{2}{10}$.

(2) For $\varsigma(k)$ as mentioned in last lemma, $d(\overline{\varsigma(k)}, \underline{\varsigma(k)_*}) \geq \frac{1}{28}$ for every k. $\blacksquare$

3.3. *Comparing Blocks*

In the next two lemmas we will consider x such that x, $\phi x \in \tilde{X}$ and let φ be a finite code approximating ϕ. We will compare how ϕ and φ map certain blocks in x and show that certain configurations are inconsistent with the existence of a finite code. Break $\{(i,j) : i, j = 1, \ldots, 9\}$ into three subsets:

$$A = \{(i,i) : i = 1,\ldots,9\},$$
$$B = \{(4,1),(5,2),(6,3),(4,7),(5,8),(6,9)\},$$

and C = the rest.

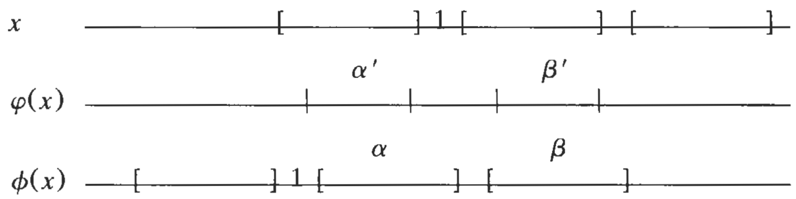
LEMMA 3.3.1. *Suppose ϕ is a measure preserving transformation and φ is a finite code such that $d(\phi x, \varphi x) < 10^{-8}$ for a.e. x. Let x be such that x, $\phi x \in \tilde{X}$. Let N be so large that*

- $|\varphi| \le 10^{-8} h_{N-2}$,
- $d(\phi x[-i, j], \varphi x[-i, j]) < 10^{-8}$, *where* $i \ge i_N$ *and* $j \ge j_N$ *where* $[-i_N, j_N]$ *denotes the indices which correspond to* $\varsigma_x(N)$, *and*
- $\phi(x) \in G_N$.

Then $(k,l)_N \notin C$.

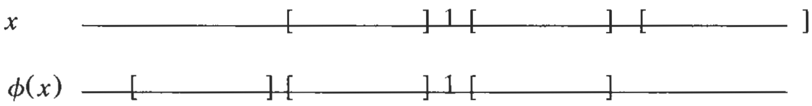
Proof. Assume $(k,l)_N$ is one of these 66 possibilities. We will show that in each case there is a block γ which appears twice in x and which maps to different blocks under ϕ which contradicts ϕ being approximable by a finite code. Only two of the 66 cases are shown here: the others are similar.

If $(k,l)_N = (1,2)$ then, x, ϕx, and φx look like



Note $|\alpha'|$ is determined by the overlap of $\varsigma_x(N)$ and $\varsigma_{\phi x}(N)$. By Lemma 3.1.1, $|\alpha'| \ge h_N/10$. Since φ is a finite code, α' and β' agree except on ends, so $d(\alpha', \beta') < 2 \cdot 10^{-8} h_{N-2}/|\alpha'| < 10^{-6}$. Since φ is within 10^{-8} of ϕ and $|\alpha|$ is large enough, $d(\alpha, \alpha') < 10^{-8} h_{N+1}/|\alpha| < 10^{-5}$ and similarly $d(\beta, \beta') < 10^{-5}$. Thus altogether we have $d(\alpha, \beta) < 10^{-5} + 10^{-5} + 10^{-6} < 10^{-4}$. Yet this contradicts Lemma 3.2.2.

Often one must go through more possibilities to reach a contradiction, as is necessary for $(k,l)_N = (1,8)$. Here the picture is



and there is no immediate contradiction, as above. Rewrite the above picture as

$$\begin{array}{l} x \sim \quad \varsigma\, 1\, \varsigma\, \varsigma \\ \phi x \sim \tau\, \tau\, 1\, \tau\, \boxed{\quad 1 \quad} \end{array}$$

and we now must consider the possibilities for box 1. These can be:

(i) $1\,\tau\,\tau\,1\,\tau$, in which case we can repeat the above argument and find a contradiction to Lemma 3.2.2.

(ii) $1\,\tau^*\,\tau^*\,1\,\tau^*$, in which case we can argue similarly to above, this time contradicting Lemma 3.2.3.

(iii) $\tau_*\,\tau_*\,1\,\tau_*$, in which case we can argue similarly to above, this time contradicting Lemma 3.2.1.

Thus the only possibility is

$$\begin{array}{l} x \sim \quad \varsigma\, 1\, \varsigma\, \varsigma \quad \boxed{\quad 2 \quad} \\ \phi x \sim \tau\, \tau\, 1\, \tau \quad \tau\, \tau\, 1\, \tau \quad \boxed{\quad 3 \quad} \end{array}$$

When we consider the possibilities for 2, any of $\varsigma\,1\,\varsigma\,\varsigma, \varsigma_*\,1\,\varsigma_*\,\varsigma_*$, or $1\,\varsigma^*\,1\,\varsigma^*\,\varsigma^*$ can appear. However, in all cases the only valid way to extend ϕx in 3 is $\tau\,\tau\,1\,\tau$. We can repeat the above step and show that the only way to avoid contradicting the Lemmas of Subsection 3.2 is to have ϕx of the form $\tau\,\tau\,1\,\tau\,\tau\,\tau\,1\,\tau\,\tau\,\tau\,1\,\tau\,\tau\,\tau\,1\,\tau$, which is impossible. ∎

LEMMA 3.3.2. *Suppose ϕ is a measure preserving transformation and φ is a finite code such that $d(\phi x, \varphi x) < 10^{-8}$ for a.e. x. Let x be such that x, $\phi x \in \tilde{X}$. Let N be so large that*

- $|\varphi| \le 10^{-8} h_{N-2}$,
- $d(\phi x[-i_N, j_N], \varphi x[-i_N, j_N]) < 10^{-8}$, *where $[-i_N, j_N]$ denotes the indices which correspond for $\varsigma_x(N)$, and*
- $\phi(x) \in G_N$.

Let $(k,l)_N \in B$. Then we cannot have $m \in \{(4,4),(5,5),(6,6)\}$ with all $n < i < m$ satisfying $(k,l)_i \in A$.

Proof. If such a situation occurred with $m = n + 1$ we would have, for instance,

$$x \sim \varsigma\,1\,\varsigma^*\,\varsigma\,1\,\varsigma^*\,1\,\varsigma^{**}\,\varsigma^*, \qquad \phi x \sim \tau\,1\,\tau\,\tau\,1\,\tau^*\,1\,\tau^*\,\tau^*.$$

This corresponds to $(k,l)_N = (4,1)$ and $(k,l)_{N+1} = (4,4)$. Since $\phi x \in G_N$, the block τ satisfies the good subblock condition and we can argue, similar to the last lemma, that a subblock γ of ς^* maps to subblocks of τ and τ^* which contradicts Lemma 3.2.1. If $m > N + 1$ then $\varsigma_x(N + 1)$ and $\varsigma_{\phi x}(N + 1)$ will be repeated (possibly) many times. Notice that ς and ς^* maintain the same proportion in these higher blocks so φ must approximate ϕ with the same accuracy as in the smaller block. We can then argue as before. ▮

3.4. *Proof of Theorem* 3.2

Let $\phi \in C(T)$ and choose φ a finite code such that $d(\phi x, \varphi x) < 10^{-8}$ for a.e. x. Fix such an x with $x, \phi x \in \tilde{X}$ and assume $\phi(x) \neq S^i T^j x$. Then by Lemma 3.1.1 there exists an infinite subset $\{n_k\}$ of the integers such that x and $\phi(x)$ lie on different n_k-blocks in their $(n_k + 1)$-blocks and these n_k-blocks overlap at least $h_{n_k}/10$.

LEMMA 3.4.1. *There cannot be infinitely many of these* n_k *with* $(k, l)_{n_k} \in C$.

Proof. Assume there is. Then we can find a large enough N and use Lemma 3.3.1 to reach a contradiction. ▮

Thus we can find an integer M such that for every $n \geq M$, either

(i) the n-blocks of x and ϕx have overlap less than $h_n/10$
(ii) the n-blocks have overlap at least $h_n/10$ and $(k, l)_n \in A \cup B$.

With only these two possibilities, notice that if the n-blocks of x and ϕx overlap at least $h_n/10$, then the $(n + 1)$-blocks overlap by at least $h_{n+1}/10$. This is because the patterns in $A \cup B$ force the continuation of a large overlap. So in fact for all large enough n, $(k, l)_n \in A \cup B$ which means that the n such that $(k, l)_n \in B$ must have density zero for x, ϕx generic points.

LEMMA 3.4.2. *There cannot be infinitely many n such that* $(k, l)_n \in B$.

Proof. Assume there is. Find N large enough to satisfy the good subblock condition. Look for $m > N$ with $(k, l)_m \in \{(4, 4), (5, 5), (6, 6)\}$. If one first finds n with $(k, l)_n \in B$, then replace N with this new index, which will also satisfy the good subblock condition. Rename it as N. Eventually one will find $m > N$ such that for $N < i < m$, $(k, l)_i \in A$. But this contradicts Lemma 3.3.2. ▮

The last two lemmas have contradicted Lemma 3.1.1. Thus there must by some (i, j) with $\phi x = S^i T^j x$. Since there is an (i, j) for a.e. x, there is some (i, j) with $\phi x = S^i T^j x$ for every $x \in V$ with $\lambda(V) > 0$. By ergodicity of T we have $\lambda(V) = 1$ and we have shown $\phi = S^i T^j$ a.e.

REFERENCES

[dJ] A. delJunco, A simple measure-preserving transformation with trivial centralizer, *Pacific J. Math.* **79**, No. 2 (1978).

[dJRS] A. delJunco, M. Rahe, and L. Swanson, Chacon's automorphism has minimal self joinings, *J. Analyse Math.* **37** (1980).

[GLL] P. Gabriel, M. Lemanczyk, and P. Liardet, Set of invariants for Anzai skew products, *Mem. Soc. Math. France (N.S.)* **47** (1991).

[K] J. King, A rank-1 transformation whose centralizer includes a Bernoulli, preprint.

[PR] K. K. Park and E. A. Robinson, The joinings within a class of Z^2 actions, *J. Analyse Math.* **57** (1991).

[R] D. J. Rudolph, An example of a measure preserving transformation with minimal self-joinings and applications, *J. Analyse Math.* **35** (1979).

[S] K. Schmidt, Dynamical systems of algebraic origins, Warwick Preprints No. **36**, 1994.