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ISOMETRIC EXTENSIONS OF ZERO ENTROPY $\mathbb{Z}^d$ LOOSELY BERNOULLI TRANSFORMATIONS

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ABSTRACT. In this paper we discuss loosely Bernoulli for $\mathbb{Z}^d$ actions. In particular, we prove that extensions of zero entropy, ergodic, loosely Bernoulli $\mathbb{Z}^d$ actions are also loosely Bernoulli.

1. INTRODUCTION AND SUMMARY OF RESULTS

In one dimension, a zero entropy transformation is loosely Bernoulli (LB) if there is one name up to the $\overline{f}$ metric. Intuitively, in one dimension this metric measures the proportion of indices between two names which can be matched in an order preserving way. In higher dimensions, the $\overline{f}$ metric measures how the relative configuration of the indices in the d-dimensional names are related. It is nontrivial to extend results to higher dimensions, due to the more complicated geometry of the definition of LB.

Many of the basic properties of one dimensional LB transformations are established in [5]. In [2] we proved that certain rank 1 $\mathbb{Z}^d$ actions are LB. The arguments used the inherent geometry of the orbits of these actions. In this paper we develop a more general "nesting" machinery than was needed in [2].

The paper is organized as follows. In section 2 we remind the reader of the higher dimensional definition of $\overline{f}$. We also define a new matching condition and discuss its relationship to the $\overline{f}$ metric. In particular, the results in this section provide some insight into the geometry of an $\overline{f}$-small permutation.

In section 3 we define properties of processes which appear weaker than LB, and prove that they are in fact equivalent to LB. This section contains the core of the new higher dimensional machinery.

In the final two sections we use our machinery to prove that k point and isometric extensions of ergodic, measure preserving and zero entropy LB $\mathbb{Z}^d$ actions are LB. Some of the arguments we provide are parallel to those in [5], but we include them to show that they can be carried through with the higher dimensional machinery. We do not assume any familiarity on the part of the reader with the results in [5].

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2. The $\overline{f}$-metric and matching names

Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space. Take T to be an ergodic, zero entropy Z^d action on $(X, \mathcal{A}, \mu)$. We can think of T as being generated by d commuting measure preserving one dimensional transformations on X, $\{T_{\vec{e}_1}, ..., T_{\vec{e}_d}\}$, where the set $\{\vec{e}_1, \ldots, \vec{e}_d\}$ is the standard basis for $\mathbb{Z}^d$. Then $T_{\vec{v}}(x) = T_{\vec{e}_1}^{v_1} \circ \cdots \circ T_{\vec{e}_d}^{v_d}(x)$, where $\vec{v} = (v_1, \ldots, v_d)$. We call $(X, \mathcal{A}, \mu), T$ a $\mathbb{Z}^d$-dynamical system. Often we will simply write (X, T).

For $n \in \mathbb{N}$, let $B_n = \{\vec{v} \in Z^d : 0 \leq v_i < n, 1 \leq i \leq d\}$. We define the ϵ-interior of B_n to be the collection of indices in B_n which are at least a distance ϵn from the boundary of B_n. For a vector $\vec{v} \in \mathbb{Z}^d$, we set $||\vec{v}||$ =max $\{|v_i| : 1 \leq i \leq d\}$.

For $n_1 < n$ we define partitions of B_n into n_1-grids. For the first grid, imagine horizontal and vertical lines in B_n drawn at the multiples of n_1, starting at $\vec{0}$. More formally, let

$$r_{\vec{0}} = \{(k_1 n_1, k_2 n_1, ..., k_d n_1) : 0 \leq k_i \leq \left[\frac{n}{n_1}\right], i = 1, ..., d\}$$

and set $R_{\vec{0}} = \{B_{n_1} + \vec{u} : \vec{u} \in r_{\vec{0}}\} \cap B_n$. We will call $R_{\vec{0}}$ an n_1-grid of B_n, the translates of B_{n_1} will be called the grid boxes, and the vectors $\vec{u} \in r_{\vec{0}}$ will be called the base points of the grid. We obtain all the n_1-grids of B_n by translating the grid $R_{\vec{0}}$ by all vectors $\vec{v} \in B_{n_1}$. We set $R_{\vec{v}} = (R_{\vec{0}} + \vec{v}) \cap B_n$ and note that $r_{\vec{v}} = r_{\vec{0}} + \vec{v}$ is the set of base points of the grid $R_{\vec{v}}$. If $C \subset B_n$, we say $R_{\vec{v}} \cap C$ is an n_1-grid of C, for any $\vec{v} \in B_{n_1}$.

Let P be a measurable, finite partition on X with label set $\{p_1, ..., p_h\}$. (T, P) is then the usual process associated with T and the partition P. For each x we define its P_n-name to be $P_n(x) : B_n \to P$ by $P_n(x)(\vec{v}) = i$ if $T_{\vec{v}}(x) \in p_i$. To simplify our notation we will call an atom of $\bigvee_{\vec{v} \in B_n} T_{\vec{v}} P$ of positive measure an n-name. The index $\vec{0}$ in an n-name will be called the base point of the name.

We start with $\pi : B_n \to B_n$, a permutation of the indices in B_n, and define a size for this permutation. This idea is defined and extended in [1] and [4].

Definition 2.1. Let $\pi : B_n \to B_n$ be a permutation of the indices of B_n. We say π *is of size* ϵ, denoted by $m(\pi) < \epsilon$, if there exists a subset S of B_n satisfying

1. $|S| > (1 - \epsilon)|B_n|$, where $|S|$ is the cardinality of the set S,
2. $||\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})|| < \epsilon||\vec{u} - \vec{v}||$ for every $\vec{u}, \vec{v} \in S$.

S is said to be the ϵ*-set* of π.

Definition 2.2. Given two P_n-names η and ξ, we define the $\overline{f}_n$*-distance* between them to be $\overline{f}_n(\eta, \xi) = \inf\ \{\epsilon > 0 :$ there exists a permutation π of B_n such that

(i) $m(\pi) < \epsilon$,
(ii) $\overline{d}(\eta \circ \pi, \xi) < \epsilon\}$.

Here $\overline{d}(\cdot, \cdot)$ denotes the Hamming metric which simply gives the proportion of locations of B_n on which the two names disagree.

Informally, we will think of π as rearranging the name η to make it $\overline{d}$ close to the name ξ, and we will often refer to π as acting on a name instead of the (technically correct) set of indices.

We now define what appears to be a matching condition with a more rigid geometric requirement. We will show that in fact this matching requirement is closely related to $\overline{f}$ matching.

Definition 2.3. For $\epsilon > 0$, $N \in \mathbb{N}$, and $0 < c < 1$, two atoms $\omega, \omega' \in \bigvee_{\vec{m} \in B_n} T_{\vec{m}} P$ are said to be (ϵ, N, c) *matchable* if there exist a permutation $\pi : B_n \to B_n$ and a set S of indices in B_n with the following properties:

1. $|S| > (1-\epsilon)|B_n|$.
2. S is the disjoint union of N-blocks, call them $\bigcup_{i=1}^{k} B_N + \vec{v}_i$.
3. π moves all the indices in S by a vector $\vec{v}$, (possibly $\vec{v} = \vec{0}$), small enough in magnitude that $i + \vec{v} \in B_n$ for all $i \in S$.
4. π moves the N blocks in S by additional amounts which can vary for each block, but are always less in magnitude than ϵn.
5. In $\omega \circ \pi$, a subset of the N-blocks is matched perfectly with N-blocks in ω'. Denote these matched blocks by $G_i, i = 1, \dots, m$. Then $|\bigcup_{i=1}^{m} G_i| > c|B_n|$. This set will be called the *matched set* G of π.
6. For $\vec{u}, \vec{v} \in S$, we have $\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < \epsilon \|\vec{u} - \vec{v}\|$.

Such a permutation π will be called an (ϵ, N, c) *permutation*, or an (ϵ, N, c) *match*. The set S will still be called the ϵ-*set* of π.

Note that if ω and ω' are (ϵ, N, c) matchable, then $\overline{f}(\omega, \omega') < 1 - c$. It is also true that $\overline{f}$ closeness implies matchability in the above sense. To make this claim precise, we will use two facts about the geometry of a small permutation. First, if $m(\pi) < \epsilon$, then $\frac{1}{\epsilon}$-blocks in the ϵ-set of π must be moved rigidly by π. The next fact requires a little more work, and the proof can be found in [4]:

Lemma 2.4 (Geometric Lemma). *Given $\epsilon > 0$, there is a $\delta > 0$ such that for all permutations $\pi : B_n \to B_n$ with $m(\pi) < \delta$, for all $\vec{v}$ in the δ-set of π,*

$$\|\pi(\vec{v}) - \vec{v}\| < \epsilon n.$$

These two facts together yield that if two n-names are $\overline{f}$-close enough then they are matchable.

Lemma 2.5. *For all $\epsilon > 0$ and $N \in \mathbb{N}$, there exist a $\delta > 0$ and an integer $n_1 > 0$ such that for all $n \geq n_1$, if $\omega, \omega' \in \bigvee_{\vec{v} \in B_n} T_{\vec{v}} P$ and $\overline{f}(\omega, \omega') < \delta$, then ω and ω' are $(\epsilon, N, 1-\epsilon)$ matchable.*

Proof. Let ϵ and N be given. Pick $\delta_1 > 0$ satisfying Lemma 2.4 with this ϵ. Let $\delta = \min\{\delta_1, \frac{\epsilon}{4N^d}\}$ and $n_1 > \frac{2dN}{\epsilon}$. Pick $n \geq n_1$ and suppose that ω, ω' are n-names with $\overline{f}(\omega, \omega') < \delta$. Let $\pi : B_n \to B_n$ be a permutation such that $m(\pi) < \delta$ and $\overline{d}(\omega \circ \pi, \omega') < \delta$. We will show that π satisfies the requirements of Definition 2.3.

If $n^d \leq \frac{1}{\delta}$, then π is the identity permutation and the result holds trivially. So suppose n is such that $n^d > \frac{1}{\delta}$. To find the set S required in Definition 2.3, we will use the portion of the δ-set of π which can be easily divided into N-blocks. To find this, first let S_1 be the δ-set of π and note that the number of indices in B_n which are not is S_1 or not matched by π is less than $2\delta|B_n|$. Then divide B_n into N-blocks, starting with the base point of the box, and let S be the union of those N-blocks which are completely contained in S_1 and completely matched by π. Thus $|S| \geq |B_n| - 2\delta|B_n|N^d - dNn^{d-1}$, so $|S| > (1-\epsilon)|B_n|$.

We also have that every $\frac{1}{\delta}$-block in S, so by our choice of δ every N-block in S, is moved rigidly by π and matched perfectly to an N-block in ω'. From the geometric lemma we are guaranteed that $\forall \vec{v} \in S$, $\|\pi(\vec{v}) - \vec{v}\| < \epsilon n$, so each N-block in S must be translated by some vector whose magnitude is less than ϵn. Thus conditions 1, 2, and 4 of Definition 2.3 are satisfied, and condition 3 is vacuously satisfied with $\vec{v} = \vec{0}$.

In this case the matched set of π is all of S, which is in the form $\bigcup G_i$ by construction, and condition 5 is satisfied with $c = 1 - \epsilon$.

Finally, for condition 6, note that $S \subset S_1$, so for every pair $\vec{u}, \vec{v} \in S_1$ we have $\|\pi\vec{u} - \pi\vec{v} - (\vec{u} - \vec{v})\| < \delta\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|$. All the conditions of $(\epsilon, N, 1 - \epsilon)$ matchability are thus satisfied. □

3. The Loosely Bernoulli Property for Zero Entropy $\mathbb{Z}^d$ Actions

Intuitively, a zero entropy loosely Bernoulli process has one name up to $\overline{f}$. Formally,

Definition 3.1. A zero entropy, ergodic process (T, P) is *loosely Bernoulli* (LB) iff for any $\epsilon > 0$ there exists an integer N_ϵ such that for any $n \geq N_\epsilon$ there is a set $W \subset \bigvee_{\vec{v} \in B_n} T_{\vec{v}}P$ such that $\mu(W) > 1 - \epsilon$ and, for ω and ω' in W,

$$\overline{f}_n(\omega, \omega') < \epsilon.$$

Definition 3.2. We say $(X, \mathcal{A}, \mu), T$ is *LB* if for every partition P of X, (T, P) is LB.

Note that to show $(X, \mathcal{A}, \mu), T$ is LB it suffices to show that the process (T, P) is LB for P a generating partition for T.

In the remainder of the section, we will define progressively weaker matching conditions which are in fact equivalent to LB. We will first state the new conditions and defer the proofs of the theorems to the end of the section.

Definition 3.3 (The Matching Condition). A zero entropy process (T, P) is said to satisfy the *matching condition* if there is a $c > 0$ such that for all $\epsilon > 0$ and $N \in \mathbb{N}$ there is an integer $n_1 > 0$ such that for all integers $n \geq n_1$, there is a set of n-names W with $\mu W > 1 - \epsilon$ and $\forall \omega, \omega' \in W$, ω and ω' are (ϵ, N, c) matchable.

If a process satisfies the matching condition then it is LB because, as the following result shows, once we can match a positive proportion of most names, we can keep matching.

Theorem 3.4. *If (T, P) is a zero entropy, ergodic, $\mathbb{Z}^d$ process satisfying the matching condition, then (T, P) is LB.*

We can, in fact, prove that a process with an even weaker matching property is LB.

Definition 3.5 (The Friendship Condition). A zero entropy process (T, P) is said to satisfy the *friendship condition* if there exist numbers $c_1, c_2 > 0$ such that for all $\epsilon > 0$ and $N \in \mathbb{N}$ there is an integer $n_1 > 0$ such that for all $n \geq n_1$ there is a set of n-names W with $\mu W > 1 - \epsilon$ and for all $\omega \in W$ there is a set $F(\omega)$ of n-names, with $\mu(F(\omega)) \geq c_2$, and for all $\omega' \in F(\omega)$ we have that ω and ω' are (ϵ, N, c_1) matchable.

We can show that a process satisfying the friendship condition is LB by arguing that if each name has a set of friends, then by sacrificing a little friendliness, we can find a large set of names which are all friendly with each other.

Theorem 3.6. *If (T, P) is a zero entropy, ergodic, $\mathbb{Z}^d$ process satisfying the friendship condition, then (T, P) satisfies the matching condition.*

Corollary 3.7. *Let (T, P) be a zero entropy, ergodic, $\mathbb{Z}^d$ process satisfying the friendship condition. Then (T, P) is LB.*

With these new results in place, showing that a process is LB will only require verifying that Definition 3.5 is satisfied.

3.1. **Proof of Theorem 3.4.** Fix $\epsilon > 0$. We will show that Definition 3.1 is satisfied with this ϵ. By hypothesis, we know that there exists a number $c > 0$ satisfying the matching condition (Definition 3.3). If $c > 1 - \epsilon$, then since (ϵ, N, c) matchability of two n-names ω, ω' implies $\overline{f}(\omega, \omega') < 1 - c < \epsilon$, we are done.

Now suppose $c < 1 - \epsilon$. It suffices to show that we can find an n_2 such that for every $n \geq n_2$ there is a set W of n-names with measure larger than $1 - \epsilon$, and all the atoms in W are (ϵ, N, c_1) matchable, with

$$c_1 = c + \frac{1}{2}c(1 - c).$$

For, if this is the case, we can keep matching until, at some stage k, $c_k > 1 - \epsilon$.

Apply the matching condition with $\frac{\epsilon^2}{100d}$ and arbitrary $N \in \mathbb{N}$ to obtain an integer n_1 and a set W_1 of n_1-names all of which are $(\frac{\epsilon^2}{100d}, N, c)$ matchable and for which

$$(1) \qquad \frac{N}{n_1} < \frac{\epsilon}{4d}, \qquad \text{and} \qquad \mu W_1 > 1 - \frac{\epsilon^2}{100d}.$$

Take $N_2 \in \mathbb{N}$ such that

$$(2) \qquad \frac{n_1}{N_2} < \frac{\epsilon^2}{2^d \cdot 4d}.$$

Apply the matching condition to $\frac{\epsilon^2}{100}$ and N_2 and apply the ergodic theorem to W_1 to obtain an integer n_2 so that for all $n \geq n_2$, we can find a set W_2 of n-names all of which are $(\frac{\epsilon^2}{100}, N_2, c)$ matchable,

$$(3) \qquad \mu W_2 > 1 - \frac{\epsilon^2}{50},$$

and

$$(4) \qquad \text{for all } x \in W_2 \qquad \frac{|\vec{v} \in B_n : T_{\vec{n}} x \in W_1|}{|B_n|} > 1 - \frac{\epsilon^2}{50}.$$

Take $\omega, \omega' \in W_2$ and let $\pi_1 : B_n \to B_n$ be an $(\frac{\epsilon^2}{100}, N_2, c)$ permutation. Let S_1 be the $\frac{\epsilon^2}{100}$-set of π_1. Consider now $\omega \circ \pi_1$ and ω'. We wish to match a subset of the indices left unmatched by π_1.

We first compute the proportion of unmatched indices of $\omega \circ \pi_1$ which

(i) lie in an N_2 block from S_1,
(ii) are in W_1, and
(iii) are such that the n_1-block based at that index is completely contained inside the unmatched N_2 block to which the index belongs.

By condition 1 of Definition 2.3, equation (4), and the fact that there are at least ϵn^d unmatched indices, the above conditions eliminate a set of indices of cardinality less than

$$\frac{\epsilon^2}{100}n^d + \frac{\epsilon^2}{50}n^d + (dn_1 N_2^{d-1})(\#\text{ of unmatched } N_2 \text{ blocks}) < \frac{\epsilon}{10}n^d.$$

Similarly, we can assume that all but an $\frac{\epsilon}{10}$ proportion of the unmatched indices in ω' will also satisfy conditions (i), (ii) and (iii).

Consider the set $\{r_{\vec{v}} : \vec{v} \in B_{n_1}\}$ of base points of all n_1-grids of B_n. For each $\vec{v} \in B_{n_1}$, let $\overline{r}_{\vec{v}}$ be that portion of $r_{\vec{v}}$ which is contained in the unmatched indices. Since $\bigcup_{\vec{v}\in B_{n_1}} \overline{r}_{\vec{v}}$ is the entire collection of unmatched indices, we can find an n_1-grid of the unmatched part of B_n such that all but $\frac{\epsilon}{5}$ of the base points of the grid boxes for both $\omega \circ \pi$ and ω' satisfy conditions (i), (ii) and (iii). Fix this grid superimposed on both $\omega \circ \pi_1$ and ω', and call a grid box with such a base point a good grid box. For a pair of good grid boxes in the same location in the two names, apply the $(\frac{\epsilon^2}{100d}, N, c)$ match guaranteed by the definition of W_1. Let π be the permutation obtained by first applying π_1, followed by the individual permutations applied to good n_1-grid boxes.

We now show that π is an (ϵ, N, c_1) permutation. Note that the j^{th} good grid box comes with an $\frac{\epsilon^2}{100d}$-set $S(j)$ and a matched set G^j consisting of matched N-blocks.

The ϵ-set S of π will consist of all the indices in S_1 except for

1. those who lie in an n_1-grid box which is not entirely contained in an N_2 box, and
2. those which are in a good grid box but
   - do not belong to $S(j)$, or
   - are in an N-block which is not contained in the $\frac{\epsilon}{4d}$-interior of its grid box.

This removes a set of indices with cardinality less than

$$dn_1N_2^{d-1}(\#\text{ unmatched } N_2 \text{ blocks}) \\ +(\frac{\epsilon^2}{100d}n_1^d + \frac{\epsilon}{4d}n_1^d + 2dNn_1^{d-1})(\#\text{ good grid boxes}),$$

which by equations (1) and (2) is less than ϵn^d. Then $|S| > (1-\epsilon)|B_n|$, so condition 1 of Definition 2.3 is satisfied.

Since, without loss of generality, we can assume N_2 and n_1 are multiples of N, condition 2 is satisfied.

Condition 3 is satisfied because the indices in S are also in S_1, the $\frac{\epsilon^2}{100}$-set of π_1, and we use for $\vec{v}$ the vector from π_1.

Note that condition 4 is automatically satisfied for the N_2 blocks moved by π_1. The bad n_1-grid boxes have no additional translation applied to them. Consider now the N-blocks in the good n_1-grid boxes. In addition to the translation by π_1, which is in magnitude less than $\frac{\epsilon^2}{100}n$, these may have been moved again by π. In particular, considering conditions 3 and 4 of Definition 2.3, their individual match will have possibly moved each by additional vectors of magnitude $< \frac{2\epsilon^2}{50d}n_1$. This is a total displacement of size less than ϵn.

For condition 5 we define the matched set G of π to be the following collection of N boxes:

(i) the matched set of π_1, i.e. the N_2 blocks matched by π_1, and
(ii) the N-blocks in $G^j \cap S$.

The collection of indices satisfying (i) is by hypothesis larger than cn^d. For (ii), note that the G^j were such that each $|G^j| > cn_1^d$. In restricting to S, we throw away less than $2d\frac{\epsilon}{4d}n_1^d + 2dNn_1^{d-1}$ indices in each good n_1-grid box in order to consider only those N blocks entirely contained in the $\frac{\epsilon}{4d}$-interior of the n_1-grid box. This is less than ϵ of each good n_1-grid box. So we have included at least $(c-\epsilon)$ proportion of each good grid box in the matched set of π. Thus of the part left unmatched by π_1 we have matched a proportion no less than

$$(c-\epsilon)(\text{proportion of unmatched indices of } \omega\circ\pi_1 \text{ in good grid boxes}).$$

Now we count the number of unmatched indices of $\omega\circ\pi_1$ in good grid boxes. An index not in a good grid box either is not in the grid we are considering, or is in a bad grid box. This is a set of indices of cardinality less than

$$2dn_1n^{d-1} + 2d(2n_1+N_2)^{d-1}n_1(\text{number of matched } N_2 \text{ boxes}) \\ + (\text{number of bad base points})n_1^d.$$

Using equation (2), the fact that there are at least ϵn^d unmatched indices, and our previous calculations about bad base points in the grid, we have that this is a proportion less than $\frac{\epsilon}{2}+\frac{\epsilon}{5}$ of the unmatched indices. Thus the proportion of unmatched indices of $\omega\circ\pi_1$ in good grid boxes is at least $1-\frac{7\epsilon}{10}$.

Putting this all together, we have matched an additional proportion of at least $(c-\epsilon)(1-\frac{7}{10}\epsilon)$. For small enough ϵ, this is larger than $\frac{1}{2}c$. So after applying the permutation π, we have matched at least $c+\frac{1}{2}c(1-c)=c_1$ proportion of the indices, satisfying condition 5 of (ϵ, N, c_1) matchability.

Finally, we show that condition 6 is satisfied. Note that every $\vec{u},\vec{v}$ in S (the ϵ-set of π) is in S_1, the $\frac{\epsilon^2}{100}$-set of π_1. In particular, if $\vec{u}$ and $\vec{v}$ are both only moved by π_1, then condition 6 holds automatically.

The only difficulty then arises if one or both of $\vec{u}$ and $\vec{v}$ in S have been moved by an individual n_1-permutation. Suppose $\vec{u}$ is such an index. There are several cases to consider:

(i) $\vec{v}$ lies in an N_2 box matched by π_1, or
(ii) $\vec{v}$ lies in a different n_1-box from $\vec{u}$, or
(iii) $\vec{v}$ lies in the same n_1-box as $\vec{u}$.

In cases (i) and (ii) we will use the fact that for such a $\vec{u},\vec{v}$,

$$\|\vec{u}-\vec{v}\| > \frac{\epsilon}{4d}n_1. \tag{5}$$

In case (i) note that $\pi(\vec{v})=\pi_1(\vec{v})$, so

$$\|\vec{u}-\vec{v}-(\pi\vec{u}-\pi\vec{v})\| \le \|\vec{u}-\vec{v}-(\pi_1\vec{u}-\pi_1\vec{v})\| + \|\pi_1\vec{u}-\pi_1\vec{v}-(\pi_2\pi_1\vec{u}-\pi_1\vec{v})\|,$$

where π_2 is the $(\frac{\epsilon^2}{100d}, N, c)$-permutation which affects $\vec{u}$. Since $\vec{u}$ and $\vec{v}$ are in S_1, the first term is less than $\frac{\epsilon^2}{100}\|\vec{u}-\vec{v}\|$. Note that condition 1 of Definition 2.3 implies that the magnitude of the vector in condition 3 is less than $\frac{\epsilon^2}{100d}n_1$. So by conditions 3 and 4 combined, π_2 will have moved $\vec{u}$ by less than $\frac{\epsilon^2}{50d}n_1$. Now by equation (5) we have

$$\|\vec{u}-\vec{v}-(\pi\vec{u}-\pi\vec{v})\| < \frac{\epsilon^2}{100}\|\vec{u}-\vec{v}\| + \frac{4\epsilon}{50}\|\vec{u}-\vec{v}\| < \epsilon\|\vec{u}-\vec{v}\|.$$

In case (ii),

$$\|\vec{u}-\vec{v}-(\pi\vec{u}-\pi\vec{v})\| \le \frac{4\epsilon^2}{100d}n_1,$$

because of conditions 3 and 4 of $\frac{\epsilon^2}{100d}$-matchability. Again by equation (5) we have

$$\|\vec{u}-\vec{v}-(\pi\vec{u}-\pi\vec{v})\| \le \epsilon\|\vec{u}-\vec{v}\|.$$

In case (iii), by construction $\vec{u}$ and $\vec{v}$ will lie in the $\frac{\epsilon^2}{100d}$-set of the same n_1-permutation. Hence condition 6 holds, and we are done.

3.2. **Proof of Theorem 3.6.** Let c_1 and c_2 be the constants from Definition 3.5, and pick $c < c_1 \cdot c_2$. We will show that the matching condition is satisfied with this c. The proof of the result for a given ϵ, N is divided into three parts. First we define the set W, then for a pair $\omega, \omega' \in W$, we define a permutation π_ω, and finally we show that π_ω is an (ϵ, N, c)-permutation.

Start by finding nonzero ϵ_1, ϵ_2 such that $c = (c_1 - \epsilon_1)(c_2 - \epsilon_2)$, and then set $\gamma = \min\{\frac{1}{d}\epsilon, \epsilon_1, \frac{1}{d}\epsilon_2\}$. Apply the friendship condition with $\frac{\gamma^2}{100}$ and an arbitrary N to find an integer n_0 large enough that

$$\frac{N}{n_0} < \frac{\gamma}{16} \tag{6}$$

and there exists a set W_{n_0} of n_0-names with $\mu(W_{n_0}) > 1 - \frac{\gamma^2}{100}$ such that every $\omega \in W_{n_0}$ has a set of friends, $F(\omega)$.

Now apply the pointwise ergodic theorem to obtain an integer n_1 large enough that there is a set U_{n_1} of n_1-names with

$$\mu(U_{n_1}) > 1 - \frac{\gamma}{100}$$

and for all $x \in U_{n_1}$ and all n_0-names ω,

$$\frac{|\vec{v} \in B_{n_1} : T_{\vec{v}}x \in \omega|}{|B_{n_1}|} \in \left(\mu(\omega) - \frac{\gamma}{100}, \mu(\omega) + \frac{\gamma}{100}\right).$$

Applying the ergodic theorem again, we obtain n_2 such that for $n \geq n_2$ there is a set of n-names W with $\mu(W) \geq 1 - \frac{\gamma}{50}$, such that for $x \in W$ we have

$$\frac{|\vec{v} \in B_n : T_{\vec{v}}x \in U_{n_1}|}{|B_n|} > 1 - \frac{\gamma}{50} \quad \text{and} \quad \frac{|\vec{v} \in B_n : T_{\vec{v}}x \in W_{n_0}|}{|B_n|} > 1 - \frac{\gamma}{50}.$$

Now take $n > n_2$ such that

$$\frac{d(n_1 + 3n_0)}{n} < \frac{\gamma}{100}, \tag{7}$$

and consider the set W as described above. We will show that W satisfies the statement of the theorem.

Take $\omega, \omega' \in W$. We will now construct an (ϵ, N, c)-permutation for this pair.

Consider those indices in ω which lie in $B_{n-(n_1+2n_0)}$. Note that by equation (7) we have

$$\frac{|B_{n-(n_1+2n_0)}|}{|B_n|} > 1 - \frac{\gamma}{100}. \tag{8}$$

Hence the proportion of $B_{n-(n_1+2n_0)}$ in ω which is not in W_{n_0} must be less than $\frac{\gamma}{25}$.

Now consider all $R_{\vec{v}}$ with $\vec{v} \in B_{n_0}$, the n_0-grids of $B_{n-(n_1+2n_0)}$, and their base points $r_{\vec{v}}$. Since $B_{n-(n_1+2n_0)} = \bigcup_{\vec{v} \in B_{n_0}} r_{\vec{v}}$, there must be a grid with all but $\frac{\gamma}{25}$ of its base points in W_{n_0}.

Fix this grid $R_{\vec{v}}$, and draw the identical grid on ω'. Note that

$$|R_{\vec{v}}| > |B_n| - d(3n_0 + n_1)n^{d-1}. \tag{9}$$

Using this with (7) and the properties of W, we have that in ω' the proportion of indices in this grid which are not locations of U_{n_1} is less than $\frac{\gamma}{25}$. Thus there is a vector $\vec{r} \in B_{n_0}$ such that for at least

$$(10) \qquad 1 - (\frac{\gamma}{25} + \frac{\gamma}{25}) = 1 - \frac{2\gamma}{25}$$

of the grid boxes:

1. in ω' this location is an occurrence of U_{n_1}, and
2. in ω this is a location in an n_0-grid box whose base point is an occurrence of W_{n_0}.

Call such an n_0-grid box in ω a "good" box, and call the n_1-name in ω' in location $\vec{r}$ its associated n_1-name. For a good grid box its associated n_1-name is at least $c_1 - \frac{\gamma}{100}$ full of its friends. Thus one location in B_{n_1}, say $\vec{m}$, is such that for $c_1 - \frac{\gamma}{100}$ of the good boxes, the location $\vec{m} + \vec{r}$ in its associated n_1-name is the base point of a friend. Let G be this subset of good boxes in ω. Then by equations (7),(9) and (10) we have

$$(11) \qquad |G| > (c_1 - \gamma)|B_n|.$$

Finally we can define π_ω. First define $\pi_1 : B_n \to B_n$ by $\pi_1(\vec{v}) = \vec{v} + \vec{r} + \vec{m}$ for $\vec{v} \in B_{n-(n_0+n_1)}$. The indices in the edges of B_n which are dislocated by the translation will be moved by π_1 to arbitrary indices vacated by the translation. Notice that after applying π_1 to ω, the n_0-names in G are lined up with a friend in ω'. Now define $\pi_\omega : B_n \to B_n$ by first applying π_1 and then, on the n_0-names in G, applying the permutation given by the $(\frac{\gamma^2}{100}, N, c_2)$ matchability of the two friends.

We now show that π_ω is the permutation that satisfies (ϵ, N, c) matchability. Recall that G consists of n_0-grid boxes in ω which line up with friends after applying π_1. Each such n_0-name has a $\frac{\gamma^2}{100}$-set S_i which is the union of N-boxes, and $|S_i| > (1 - \frac{\gamma^2}{100})|B_{n_0}|$. Let $\tilde{S}_i$ be the union of those N-boxes which lie entirely in the $\frac{\gamma}{16}$ interior of B_{n_0}. We have thus eliminated at most $2d(\frac{\gamma}{16}n_0 + N)n_0^{d-1}$ indices from S_i, which by (6) is less than $\frac{d\gamma}{4}n_0^d$. Thus $|\tilde{S}_i| > (1 - \frac{\gamma^2}{100} - \frac{d\gamma}{4})|B_{n_0}|$. Now set S to be the union of these $\tilde{S}_i$ plus the n_0-boxes of the grid not in G. Then by the above calculation, equation (8), and our choice of γ, we have $|S| > (1-\epsilon)|B_n|$. This gives us condition 1 of Definition 2.3.

Now take $\vec{u}, \vec{v} \in S$. If $\vec{u}, \vec{v}$ lie in the same n_0-box, by construction

$$\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < \frac{\gamma^2}{100}\|\vec{u} - \vec{v}\| < \epsilon\|\vec{u} - \vec{v}\|.$$

Otherwise, suppose $\vec{u}, \vec{v}$ are from distinct n_0-boxes. If both are n_0-boxes not in G, then

$$\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| = 0.$$

If at least one n_0-box is in G, we have from $(\frac{\gamma^2}{100}, N, c_2)$ matchability that $\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < 2\frac{\gamma^2}{100}n_0$. Because we are only considering the $\frac{\gamma}{16}$ interior of the n_0-boxes in G, we also have $\|\vec{u} - \vec{v}\| \geq \frac{\gamma}{16}n_0$. So

$$\|\pi(\vec{u}) - \pi(\vec{v}) - (\vec{u} - \vec{v})\| < \epsilon\|\vec{u} - \vec{v}\|,$$

and condition 6 is satisfied.

Without loss of generality we can assume n_0 is a multiple of N, and thus condition 2 is also satisfied.

Next notice that all the indices in S were first shifted by the vector $\vec{m}+\vec{r}$ satisfying $\|\vec{m}+\vec{r}\| < n_0+n_1$. Since S is contained in the subbox of B_n which is a distance n_0+n_1 from the edge of B_n, condition 3 of Definition 2.3 is satisfied.

The permutation π further moves the N-boxes found in G by amounts given by the $(\frac{\gamma^2}{100}, N, c_2)$ matchability of their respective n_0-boxes. Their total translation is thus less than $n_0 + \frac{\gamma^2}{100}n_0$, which by (7) and our choice of γ is less than ϵn, as required for condition 4.

Recall that S_i was the $\frac{\gamma^2}{100}$-set of a good box and $\tilde{S}_i \subset S_i$ was the subset of S_i of N-blocks which were entirely in the $\frac{\gamma}{16}$-interior of B_{n_0}. Let $\bigcup G_i$ be the union of those N-blocks in all the $\tilde{S}_i$'s which were matched perfectly by the $(\frac{\gamma^2}{100}, N, c_2)$ matchability. Then $|\bigcup G_i| > (c_2 - \frac{d\gamma}{4})|G|$, and hence, by (11) and our choice of γ, $|\bigcup G_i| > (c_2 - \frac{d\gamma}{4})(c_1-\gamma)|B_n| > c|B_n|$.

Since conditions 1 through 6 of Definition 2.3 are satisfied, we have that $\omega, \omega' \in \hat{W}$ are (ϵ, N, c) matchable, as wanted.

4. k POINT EXTENSIONS

Let $k \geq 2$ be an integer. To define a k-point extension of the ergodic, zero entropy $\mathbb{Z}^d$ action (X,μ,T) we let $\{c_1,\ldots,c_k\}$ denote the discrete space with k points. We think of each c_i as representing a different color. Let $\overline{X} = X \times \{c_1,\ldots,c_k\}$, $\overline{\mathcal{A}} = \mathcal{A} \times 2^{\{c_1,\ldots,c_k\}}$, and $\overline{\mu}(A\times\{c\}) = \frac{1}{k}\mu(A)$, where $A \in \mathcal{A}$ and $c \in \{c_1,\ldots,c_k\}$.

Let S_k denote the symmetric group on k symbols. Let $h : X \times \mathbb{Z}^d \to S_k$ be a measurable T cocycle. So for every $\vec{m},\vec{n} \in \mathbb{Z}^d$ we have

$$h(x,\vec{m}+\vec{n}) = h(x,\vec{m}) \circ h(T_{\vec{m}}(x),\vec{n}).$$

We can then define a $\mathbb{Z}^d$-action $\{T^h_{\vec{v}}\}$ in the following way:

$$T^h_{\vec{v}}(x,c) = (T_{\vec{v}}(x), h_{\vec{v}}(x)c).$$

For any such extension, if T has zero entropy then T^h also has zero entropy. If P_1 is a generating partition for T, then $P = P_1 \vee \{\{c_1\},\ldots,\{c_k\}\}$ is a generating partition for T^h. We call it the extension of P_1. For the remainder of this section we fix P_1, a generating partition for T, and set P as above.

For $1 \leq i \leq d$ we denote the measurable functions $h : X \times \{\vec{e}_i\}$ by h_i. We say the extension $(\overline{X},\overline{\mathcal{A}},\overline{\mu}), T^h$ is trivial if h_i is constant for every i.

In this section we will first show that if (T^h,P) is an ergodic trivial extension of a zero entropy LB system, then (T^h,P) is LB. We will then prove the result for a non-trivial k-point extension by reducing the argument to the trivial case.

Theorem 4.1. *If T is an ergodic, zero entropy and LB $\mathbb{Z}^d$ action, and T^h is an ergodic trivial k-point extension of T, then T^h is LB.*

Proof. Let P be the extension of a generating partition of T. Since P is a generating partition for T^h, by Corollary 3.7 it suffices to show that (T^h,P) satisfies the friendship condition. We will show that (T^h,P) satisfies Definition 3.5 with $c_1 = c_2 = \frac{1}{2k}$.

Let $0 < \epsilon < \frac{1}{2k}$ and $N \in \mathbb{N}$ be fixed. Find $\delta < \epsilon$ and $n_1 \in \mathbb{N}$ satisfying Lemma 2.5. Since (X,T) is LB, we can assume n_1 is such that for every $n > n_1$ and δ-a.e. n-names ω and ω', $\overline{f}_n(\omega,\omega') < \delta$. Let this set of atoms be the set W_n, and put $\overline{W}_n = \{\overline{\omega} : \overline{\omega}$ is an extension of $\omega \in W_n\}$. For $\overline{\omega} \in \overline{W}_n$, we will construct a set

$F(\overline{\omega})$. We will do this by showing that every other atom in W_n has an extension which belongs to $F(\overline{\omega})$. This will be done by first finding the proposed extension, showing that the collection of such things has sufficient measure, and then showing that indeed such extensions can be matched to $\overline{\omega}$.

So fix $\overline{\omega}$, an extension of ω, and let ω' be another point in W_n. By Lemma 2.5 we have that ω, ω' are $(\epsilon, N, 1-\epsilon)$ matchable. Let π be the permutation satisfying the matchability conditions.

Consider G, the matched set of π. Say $G = \bigcup G_i$, where each G_i is a matched N-block. Denote their counterparts in ω' by $\{G_i'\}$ and the corresponding colored boxes in $\overline{\omega}$ by $\{\overline{G}_i\}$. We know that ω' has exactly k extensions and the color at every index is different in different extensions. Call the extensions $\overline{\omega}_j'$ for $j = 1, \ldots, k$. We denote the corresponding colorings of G_i' by $\overline{G}_{i,j}'$.

Look at the color at the lower left hand corner of box $\overline{G}_i$ in $\overline{\omega}$. In one of the $\overline{G}_{i,j}'$, the color at the lower left hand corner must be the same as in $\overline{G}_i$; hence the entire N-box must have the same coloring as in $\overline{G}_i$. This is true for every i, so there is a $j \in \{1, \ldots, k\}$ such that for at least $\frac{1}{k}$ of the boxes $\overline{G}_i$, π matches $\overline{G}_i$ perfectly with $\overline{G}_{i,j}'$ in $\overline{\omega}_j'$. Call such an extension of ω' a good extension, and set $F(\overline{\omega}) = \{\overline{\omega}' \in \overline{W}_n : \overline{\omega}' \text{ is a good extension of } \omega' \in W_n\}$.

By the argument above it is clear that

$$\overline{\mu}(F(\overline{\omega})) \geq \frac{1}{k}\overline{\mu}(\overline{W}_n) > \frac{1}{k}(1-\delta) > \frac{1}{k} - \frac{\epsilon}{k} > \frac{1}{2k},$$

as desired.

We now want to show that, for $\overline{\omega}' \in F(\overline{\omega})$, $\overline{\omega}$ and $\overline{\omega}'$ are $(\epsilon, N, \frac{1}{2k})$ matchable.

Let S and π be as defined by the matchability on the base space, so all conditions of Definition 2.3 except condition 5 are automatically satisfied. To see that condition 5 holds, note that the indices in $\overline{G}$, the matched set of π applied to the extension, is a subset of the indices in G, the original matched set. Our earlier argument shows that

$$|\overline{G}| \geq \frac{1}{k}|G| > \frac{1}{k}(1-\epsilon)|B_n|,$$

so condition 5 is satisfied with $c = \frac{1}{2k}$. □

Theorem 4.2. *If T is an ergodic, zero entropy and LB $\mathbb{Z}^d$ action, and T^h is an ergodic k-point extension of T, then T^h is LB.*

Proof. Let P_1 be a generating partition for T, and for $1 \leq i \leq d$ let $E_i = \{h_{\vec{e}_i}^{-1}(\sigma)\}_{\sigma \in S_k}$. The partition $P_2 = P_1 \vee E_1 \vee \ldots \vee E_d$ is also a measurable, generating partition for T; hence its extension P is generating for T^h. Since T is LB, then (T, P_2) is also LB.

Note that with the partition P, every P_2-name ω has exactly k extensions, the color at every index is different for different extensions, and knowing the color of one index determines the color of a whole box. In fact, with this partition we can treat the extension as though it were a trivial extension. The argument now follows as in the proof of Theorem 4.1. □

5. Isometric extensions

In this section we let (C, ρ) be a compact, homogeneous metric space and G be the group of all isometries of C. Note that G is then a compact group [3]. Let m be the G-invariant measure on C, and (X,μ,T) a free, measure preserving, ergodic, zero entropy $\mathbb{Z}^d$ action. Suppose $h : X \times \mathbb{Z}^d \to G$ is a measurable T cocycle. Namely, for all $\vec{n},\vec{m} \in \mathbb{Z}^d$ we have that

$$h(x,\vec{n}+\vec{m}) = h(x,\vec{n}) \circ h(T_{\vec{n}}x,\vec{m}). \tag{12}$$

If for $\vec{n} \in \mathbb{Z}^d$ we define $T^h : X \times C \to X \times C$ by

$$T^h_{\vec{n}}(x,c) = (T_{\vec{n}}x, h(x,\vec{n})(c)),$$

then by equation (12), T^h will be a measure preserving $\mathbb{Z}^d$ action on $(X \times C, \mu \times m)$. T^h will have the same entropy as T [3]. We will refer to T^h as a compact group extension of T.

Theorem 5.1. *Let (X,μ,T) be a free, ergodic and measure preserving, zero entropy, and LB $\mathbb{Z}^d$ action. Let (C,ρ) be a compact, homogeneous metric space and G be the group of all isometries of C. Let h: $X \times \mathbb{Z}^d \to G$ be a T cocycle, and suppose T^h is an ergodic isometric extension of T. Then T^h is LB.*

Proof. Let T and h be as above. We abbreviate $h_i(x) = h(x,\vec{e}_i)$ for each dimension $1 \le i \le d$. Denote $\mu \times m$ by $\hat{\mu}$. Take a partition P of X which is a generating partition for T, and a partition Q on C such that the topological boundary of Q has measure zero, and such that $P \times Q$ is a generating partition for T^h. We now show that $(T^h, P \times Q)$ is LB. Since T is zero entropy so is T^h. Thus we need to prove that $(T^h, P \times Q)$ satisfies Definition 3.1.

We will do two simplifications: we will approximate Q by a simpler partition R, and we will approximate each h_i by a step function. The outline of the proof is as follows. We will initially use the partition R and a weakened sense of matching of $P^L \times R$ names to find modified friends for a large set of atoms, where P^L is a refinement of P to be determined later and where we will later make the term "modified friends" precise. We will then use our previous machinery to proceed to a large set of atoms all of which are modified friends, and then to conclude a modified form of LB. Finally, we will remove this modification and prove that T^h is LB.

Fix $\epsilon > 0$ and $N \in \mathbb{N}$. Suppose R is a partition of the space C into r elements, and label the sets of R initially in arbitrary order by $\{R_1, R_2, \ldots, R_r\}$. Let

$$d_1 = \min_{i\in\{1,\ldots,r\}} \{diam(R_i)\}, \qquad d_2 = \max_{i\in\{1,\ldots,r\}} \{diam(R_i)\},$$
$$\beta_1 = \min_{i\in\{1,\ldots,r\}} \{m(R_i)\}, \qquad \beta_2 = \max_{i\in\{1,\ldots,r\}} \{m(R_i)\}.$$

For the alphabet of the partition R, construct a labeling scheme so that by reading the label assigned to set R_i we can identify all sets R_j such that

$$\rho(R_i,R_j) \le 2d_2. \tag{13}$$

We will say that two sets R_i, R_j which satisfy (13) are adjacent.

Let

$$\epsilon_1 < \min\left\{\frac{\epsilon}{12N^d}, \frac{d_2}{Nd}\right\} \tag{14}$$

and choose $L > 0$ so large that there are functions $\tilde{h}_i : X \to G$ such that

1. if $\omega \in \bigvee_{\vec{v}\in B_L} T_{\vec{v}}P$, then $\tilde{h}_i(x) = g_i$ for every $x \in \omega$, $1 \le i \le d$, and
2. for ϵ_1-almost every $x \in X$, $\rho(h_i(x), \tilde{h}_i(x)) < \epsilon_1$, $1 \le i \le d$.

We will now use the partition $P^L = \bigvee_{\vec{v} \in B_L} T_{\vec{v}} P$ instead of P. Since P^L is a refinement of P, $P^L \times Q$ is also a generating partition for T^h and it suffices to show LB for this partition.

Let $B = \{x : \rho(h_i(x), \tilde{h}_i(x)) > \epsilon_1$ for some $1 \le i \le d\}$. Then $\mu(B) < \epsilon_1$.

Applying loosely Bernoulli and the ergodic theorem to the base, we obtain $n > 0$ and a set of atoms W_n such that

$$\mu(W_n) > 1 - \frac{\epsilon}{3}, \tag{15}$$

$$\omega, \omega' \in W_n \text{ are } \left(\frac{\epsilon}{3}, N, 1 - \frac{\epsilon}{3}\right) \text{ matchable,} \tag{16}$$

and the set $H = \{x : |\vec{v} \in B_n : T_{\vec{v}} x \in B| / |B_n| < 2\epsilon_1\}$ has measure

$$\mu H > 1 - \frac{\epsilon^2}{18} \frac{\beta_1}{4\beta_2}. \tag{17}$$

Next, consider the atoms of W_n. We will call an atom ω "good" if

$$\mu(\omega \cap H^c) < \frac{1}{4} \frac{\beta_1}{\beta_2} \frac{\epsilon}{6} \mu(\omega). \tag{18}$$

Consider only those atoms in W_n which are good, and still call this new set W_n. Then

$$\mu(W_n) > 1 - \frac{\epsilon}{3} - \frac{\epsilon}{3}. \tag{19}$$

Fix an atom ω from W_n and consider all the $P^L \times R, n$-names which have ω as their P^L label. We call these the extension atoms of ω. We want to consider only those extension atoms which are substantially covered by the set $H \times C$. We define an extension-atom $\hat{\omega}$ to be "bad" if $\hat{\mu}(\hat{\omega} \cap (H^c \times C)) > \frac{1}{2} \hat{\mu}(\hat{\omega})$. Since every $\omega \in W_n$ is a good atom in the base space, we have that for a fixed good base atom ω, the measure of all the bad extension atoms of ω is less than $\frac{\epsilon}{3} \mu(\omega)$.

Let $\hat{W}_n = \bigcup_{\omega \in W_n} \{\hat{\omega} : \hat{\omega}$ is a good extension of $\omega\}$. Then

$$\mu(\hat{W}_n) > (1 - \frac{\epsilon}{3})(1 - \frac{\epsilon}{3} - \frac{\epsilon}{3}) > (1 - \epsilon).$$

For the rest of our work with $P^L \times R$ names we will modify our notion of matching. Let $\hat{\omega}$ and $\hat{\omega}'$ be two $P^L \times R$ names. We will say that the R-labels i and j at any index $\vec{v}$ in $\hat{\omega}$ and $\hat{\omega}'$ agree if equation (13) is satisfied for this R_i and R_j.

We now fix $\hat{\omega} \in \hat{W}_n$ and let ω be its P^L labeling. We will find a set of friends for $\hat{\omega}$ with this new notion of matchability. Consider any atom ω' in W_n. By our construction of W_n we know that there is an $(\frac{\epsilon}{3}, N, 1 - \frac{\epsilon}{3})$ match $\pi : B_n \to B_n$ between ω and ω'. Thus we know most of the indices of ω can be divided into N-blocks so that π matches most of these blocks perfectly to N-blocks in ω'.

By our construction of W_n, we know we can find $x \in \omega$ and $x' \in \omega'$ such that $x, x' \in H$. We wish to remove from consideration those matched N-blocks which, in the orbit of either x or x', contain an occurrence of the set B. By the definition of H we know that there are less than $2\epsilon_1 n^d$ occurrences of B in either block of the orbit; hence we will have thrown away at most $4\epsilon_1 n^d$ N-blocks. By equation (14) this is a set of indices of proportion less than $\frac{\epsilon}{3}$ of an n-name.

Call the first lexicographic index of any N-box its base point. Consider one of the remaining N-boxes in $\hat{\omega}$, and suppose the R-labeling at $\vec{v}$, its base point, is i. We claim the following: of the extension atoms of ω' which lie in $\hat{W}_n$, those which see the R-label i at index $\vec{\pi}(v)$ have measure at least $\frac{\beta_1}{2}$ and at most $\frac{\beta_2}{2}$. To see why, note that at least β_1 and at most β_2 of the set of all extension atoms of ω' have R-label i at index $\pi(\vec{v})$. Of the set of points in this collection of extension atoms, at most $\beta_2\mu(\omega' \cap H^c)$ are in $H^c \times C$. We are considering only those extensions which are at least half covered by H, so we remove from consideration a collection of atoms of measure at most $2 \times \beta_2\mu(\omega' \cap H^c)$, which is less than $2\beta_2\frac{1}{4}\frac{\beta_1}{\beta_2}\epsilon\mu(\omega')$. This leaves us with a collection of extension atoms of measure at least $\beta_1\mu(\omega') - \frac{\beta_1}{2}\epsilon\mu(\omega') > \frac{\beta_1}{2}\mu(\omega')$.

So, for each such base point $\vec{v}$ in ω, and for any other $\omega' \in W_n$, at least $\frac{\beta_1}{2}\mu(\omega')$ of the extension atoms of ω' have R-labels at $\pi(\vec{v})$ which agree with the R-label $\vec{v}$ in ω. Thus for at least $\frac{\beta_1}{4}\mu(\omega')$ of the extensions of ω', the R-label must agree with $\hat{\omega}$ at the base point of at least $\frac{\beta_1}{4}$ of the remaining N-boxes of ω.

Take these extensions over all atoms $\omega' \in W_n$, and call this set $\hat{F}(\hat{\omega})$. Note that by the above calculation and condition (19) we have

$$\mu(\hat{F}(\hat{\omega})) > \frac{\beta_1}{4}\mu(W_n) > \frac{\beta_1}{4}(1-\epsilon) > \frac{\beta_1}{8}. \tag{20}$$

We now argue that for all $\hat{\omega}' \in \hat{F}(\hat{\omega})$, $\hat{\omega}$ and $\hat{\omega}'$ are friends in the modified sense of matching. Fix $\hat{\omega}' \in \hat{F}(\hat{\omega})$ and consider an N-box in $\hat{\omega} \circ \pi$ for which the P^L-label agrees with the N-box at the same location in $\hat{\omega}'$ and the R-label at $\vec{v}$ and $\pi(\vec{v})$ (the respective base points of the N boxes) is the same. Let x and x' be as before.

Since the P^L-labels of the N-boxes agree, the points x and x' visit the same P^L elements in those pieces of their orbit. Hence for every $\vec{m} = (m_1, ..., m_d) \in B_N$ and $i = 1, \ldots, d$ we have

$$\tilde{h}_i(T^{\vec{v}+\vec{m}}x) = \tilde{h}_i(T^{\pi(\vec{v})+\vec{m}}x'). \tag{21}$$

Recall that we are only considering N-blocks which don't contain an occurrence of B in either orbit. Further, by our choice of L we have, for each $\vec{m} = (m_1, \cdots, m_d) \in B_N$,

$$|h(T^{\vec{v}}x, \vec{m}) - \tilde{h}_1(T^{\vec{v}}x) - \ldots - \tilde{h}_1(T^{\vec{v}+m_1\vec{e}_1+\ldots+m_{d-1}\vec{e}_{d-1}}x) - \ldots - \tilde{h}_d(T^{\vec{v}+m_1\vec{e}_1+\ldots+(m_d-1)\vec{e}_d}x)| \leq \epsilon_1 dN, \tag{22}$$

and the same is true for x'. So for any $\vec{m} \in B_N$ the difference in the rotations given by $h(x, \vec{m}+\vec{v})$ and $h(x', \vec{m}+\pi(\vec{v}))$ cannot differ by more than $2\epsilon_1 dN$. By equation (14) this is less than d_2. So, if the R-label at $\vec{v}$ and $\pi(\vec{v})$ is the same in $\hat{\omega}$ and $\hat{\omega}'$, then the R-labels at $\vec{v}+\vec{m}$ in $\hat{\omega}$ and $\pi(\vec{v})+\vec{m}$ in $\hat{\omega}'$ must match in the modified sense. Hence, the R-label of the entire N-block in $\hat{\omega} \circ \pi$ must match in the modified sense that in $\hat{\omega}'$.

Thus, in the modified sense, we have $(\frac{\epsilon}{3}, N, \frac{\beta_1}{4}(1-\frac{\epsilon}{3}-\frac{\epsilon}{3}))$ matchability between $\hat{\omega}$ and $\hat{\omega}'$, and T^h satisfies a modified friendship condition with $c_1 = c_2 = \frac{\beta_1}{8}$. We can argue, as in Corollary 3.7, that this will yield a modified version of LB.

We now argue that $(T^h, P \times Q)$ satisfies Definition 3.1.

Fix $\epsilon > 0$. By our above comments and a modified version of Lemma 2.5 we know we can find $m > 0$ such that for all $n > m$ there is a set $\hat{W}_n$ of $P^L \times R$ atoms with $\hat{\mu}W_n > 1 - \frac{\epsilon}{2}$, and all $\hat{\omega}, \hat{\omega}' \in \hat{W}_n$ are (modified) $\overline{f}_n$-close enough that they are $(\frac{\epsilon}{2}, N, 1-\frac{\epsilon}{2})$ modified matchable.

Suppose that the partition R was constructed so that β_2 in our above discussion was small enough that if $\mathcal{A}$ is the collection of sets in R which either intersect the boundary of Q or lie adjacent to a set which intersects the boundary of Q, then $\mathcal{A}$ has measure less than $\frac{\epsilon}{8N^d}$. Let

$$\hat{C} = \left\{(x,\theta) : \frac{|\vec{v} \in B_n : T^h_{\vec{v}}(x,\theta) \in X \times \mathcal{A}|}{|B_n|} < \frac{\epsilon}{4N^d}\right\},$$

and using the ergodic theorem assume m is large enough so that, for all $n > m$, $\hat{\mu}\hat{C} > 1 - \frac{\epsilon}{2}$. Now throw away from $\hat{W}_n$ any atom $\hat{\omega}$ for which $\hat{\mu}(\hat{\omega} \cap \hat{C}) = 0$. Call the remaining set $\hat{W}_n$, and note that $\hat{\mu}\hat{W}_n > 1 - \epsilon$.

Fix $\hat{\omega}$ and $\hat{\omega}'$ in $\hat{W}_n$, and $(x,\theta) \in \hat{\omega}$ and $(x',\theta') \in \hat{\omega}'$ such that $(x,\theta),(x',\theta') \in \hat{C}$. By the definition of C we know that by throwing away at most $\frac{\epsilon}{2N^d}|B_n|$ N-boxes, we can guarantee that each modified matched N-box does not visit a set from the collection $\mathcal{A}$ along the orbit of x and x'. Note that this eliminates a set of indices $< \frac{\epsilon}{2}|B_n|$.

On the remaining N-blocks it now follows that each R-set corresponds to a unique element of the partition Q. Hence we can, in a well defined manner, erase the R-labels and replace them by Q labels. Further, by equation (13) any two sets whose labels agree in the modified sense and do not belong to $\mathcal{A}$ must lie in the same Q element. Hence, on the remaining N-boxes in $\hat{\omega}$ and $\hat{\omega}'$, the modified R-label match translates to an actual Q-label match.

It follows then that the (P^L, Q) names are $(\frac{\epsilon}{2}, N, 1-\epsilon)$ matchable. Hence $\overline{f}_n(\hat{\omega},\hat{\omega}') < \epsilon$, and we are done. □

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