TOPOLOGY
PROCEEDINGS
Volume 22, 1997

IP CLUSTER POINTS, IDEMPOTENTS, AND RECURRENT SEQUENCES

Kamel N. Haddad and Aimee S. A. Johnson

Abstract

We consider the dynamical system (M, S) where M is the orbit closure of a nonperiodic recurrent sequence of 0's and 1's (for example, the Morse sequence) and S is the shift map. The enveloping semigroup is $E(M) = \overline{\{S^n : n \in \mathbf{Z}\}}$, where the closure is taken in the topology of pointwise convergence. H. Furstenberg was the first to establish the existence of relationships between recurrence, IP sets, and idempotents in the enveloping semigroup, and the first author has proven that the closure of the set of idempotents coincides with the IP cluster points. In this paper the authors compute this set for (M, S) and shed light on other combinatorial properties of generalized Morse sequences.

1 Introduction

An IP subset of **N** (or **Z**) is one which coincides with the set of finite sums taken from an infinite sequence $(p_n)_{n=1}^{\infty}$ of distinct elements in **N** (or **Z**). The notion of IP sets gained interest in combinatorial mathematics when Hindman proved in 1974 that if $\mathbf{N} = C_1 \cup \ldots \cup C_k$, then there exists an index j such that C_j contains an IP set [Hi]; a theorem first conjectured by Graham and Rothschild [GR]. This prompted Furstenberg to investigate the relationship between IP sets, recurrence, and idempotents in the enveloping semigroup of a dynamical system [F]. Several papers followed in this new vein, most notably by Auslander and Furstenberg [AF], Berend [B], Bergelson and Hindman [BH], and Furstenberg and Weiss [FW1], [FW2].

In this paper, we investigate connections linking dynamical systems with combinatorial number theory for a special class of spaces which are subshifts of the full shift on two symbols and include the space consisting of the orbit closure of the standard Morse sequence in $\{0,1\}^{\mathbf{Z}}$ [M].

If X is a compact topological space and $T : X \to X$ is continuous, we will call the dynamical system (X, T) an **N**-cascade when we allow only positive iterations of T and a **Z**-cascade, if T is a homeomorphism and both positive and negative iterations of T are allowed. Consider X^X, the space of all functions from X to itself, equipped with the product topology, and define $E(X) = \overline{\{T^n : n \in \mathbf{Z}\}}$, a topological subspace of X^X. $E(X)$ is a compact semigroup in X^X, called the enveloping semigroup of X [E1]. If we put $H(X) = E(X) - \{$isolated points of $E(X)\}$, then $H(X)$ is also a compact semigroup in X^X [H, proposition 3.1]. A well known theorem by Ellis (see for instance [E2], corollary 2.10) states that a nonempty compact semitopological semigroup must contain idempotents. Thus $H(X)$ contains idempotents, and we will denote the set of idempotents in $H(X)$ by $J(X)$.

Definition 1.1 *An IP set in* $\mathbf{N}$ *(or* $\mathbf{Z}$*) is a subset* P *of* $\mathbf{N}$ *(or* $\mathbf{Z}$*) which coincides with the set of finite sums* $p_{n_1} + \ldots + p_{n_k}$, $n_1 < \ldots < n_k$, *taken from a sequence* $(p_n)_{n=1}^{\infty}$ *of distinct elements in* $\mathbf{N}$ *(or* $\mathbf{Z}$*). The sequence* $(p_n)_{n=1}^{\infty}$ *is called the generating sequence of* P.

In [H], the first author introduced the notion of an IP cluster point in $H(X)$ along an IP set P to be the following:

Definition 1.2 *For a* $\mathbf{N}$ *(or* $\mathbf{Z}$*) cascade, an element* f *in* $H(X)$ *is an IP cluster point (IPCP) along an IP subset* P *of* $\mathbf{N}$ *(or* $\mathbf{Z}$*) if given a neighborhood* U *of* f *in* X^X*, the set* $\{n \in P : T^n \in U\}$ *contains an IP subset of* P.

Let $I(X)$ denote the set of IP cluster points along $\mathbf{N}$ (or $\mathbf{Z}$). In [H] is a study of IP cluster points, from which we need two results: 1) $I(X)$ is never empty [H, theorem 3.1] (the proof involves a standard compactness argument) and 2) $I(X) = J(X)$ for all compact systems (X,T) [H, corollary 3.2]. This explains our interest in the study of IP cluster points.

We will compute $J(X)$ and $I(X)$ when X is the orbit closure under the shift map of a nonperiodic recurrent sequence (in the sense of Keane), and show that in every case, $J(X) = I(X)$, a set with finite cardinality.

The authors would like to thank K. Berg and H. Furstenberg for many helpful discussions during the preparation of this paper, and the referee for numerous helpful comments.

2 Preliminaries

In [K], Keane introduced the following notation to express infinite sequences of zeros and ones: A sequence $b = [b_0 b_1 \ldots b_m]$ of zeros and ones is called a block, whose length $m+1$ will be denoted by $l(b)$. The block obtained from b by interchanging the zeros and ones is called the dual of b and is denoted by $\overline{b}$. We can define an operation "$\times$" on blocks by first setting $b \times 0 = b$, $b \times 1 = \overline{b}$ and then for a fixed block $c = [c_0 c_1 \ldots c_n]$, defining $b \times c$ to be $b \times c_0 + b \times c_1 + \ldots + b \times c_n$, where "+" denotes concatenation. If $c_0 = 0$, then $b \times c$ is simply an extension of b. The operation $\times$ just defined is associative, and if $b^0, b^1, b^2, \ldots$ are chosen so that $b_0^i = 0$ and $l(b^i) > 1$ for every $i \geq 1$, then the sequence $b^0 \times b^1 \times b^2 \times \ldots$ is well defined and infinite. Such a sequence shall be called a recurrent sequence. It is shown in [K, lemma 1] that a recurrent sequence is periodic if and only if there exists a $k \in \mathbf{N} \cup \{0\}$ such that $b^k \times b^{k+1} \times \ldots$ equals either $[00000\ldots]$ or $[010101\ldots]$.

Denote $\{0,1\}^{\mathbf{Z}}$ by Ω and $\{0,1\}^{\mathbf{N}}$ by X. Let S be the shift map defined on Ω or X by $S(x)(n) = x(n+1)$ for $x \in \Omega$ or X and $n \in \mathbf{Z}$ or $\mathbf{N}$. Let $\mathcal{B}$ be the set of finite blocks of 0's and 1's, and for z in Ω or X, if $a \in \mathbf{Z}$ and $b \in \mathbf{N}$, let $z(a,b) = [z_a z_{a+1} \ldots z_{a+b-1}]$. Let $\mathcal{B}_z \subseteq \mathcal{B}$ be the set of all finite blocks occurring in z:

$$\mathcal{B}_z = \{b \in \mathcal{B} : \exists k \in \mathbf{Z} \text{ such that } z(k, l(b)) = b\}.$$

The orbit $\{S^n \omega : n \in \mathbf{Z}\}$ of a point $\omega \in \Omega$ will be denoted by $O(\omega)$. For each $x \in X$, we define the subset O_x of Ω by setting:

$$O_x = \{\omega \in \Omega : \mathcal{B}_\omega \subseteq \mathcal{B}_x\}.$$

It is shown in [K, lemma 4] that if x is a recurrent sequence, then there exists a point $\omega \in O_x$ with $[\omega_0 \omega_1 \omega_2 \ldots] = x$. Keane constructs ω in the following manner:

Let $x = b^0 \times b^1 \times b^2 \ldots$, and for any $b = [b_0 b_1 \ldots b_n] \in \mathcal{B}$, set $\hat{b} = [b_n \ldots b_1 b_0]$.

For $i \in \mathbf{N} \cup \{0\}$, define

$$d^i = \begin{cases} b^i & \text{if } b^i_{l(b^i)-1} = 0 \\ \overline{b^i} & \text{otherwise} \end{cases}$$

Then $\hat{x} := \hat{d^0} \times \hat{d^1} \times \hat{d^2} \times \ldots$ is well defined and belongs to X. Now define ω by

$$[\omega_0\omega_1\ldots] := x$$
$$[\omega_{-1}\omega_{-2}\ldots] := \hat{x}.$$

Keane goes on to show that ω thus defined is indeed in O_x. A similar argument shows that the point constructed by using the dual of the entries at the negative subscripts of ω is also in O_x. We write the pertinent information in the following two corollaries:

Corollary 2.1 *Let x be a recurrent sequence and $\omega \in O_x$ with $x = [\omega_0\omega_1\ldots]$. Let $\nu = [\ldots\overline{\omega_{-2}\omega_{-1}}\omega_0\omega_1\ldots]$. Then $\nu \in O_x$, and ω and ν are the only elements of O_x which agree with x at all non-negatively indexed positions.*

Thus a recurrent sequence x can be extended to the left in two and only two ways so that the extension is in O_x. Furthermore, the two left extensions are dual of each other. Note that ω and ν are positively asymptotic, whereas ω and $\overline{\nu}$ are negatively asymptotic.

Corollary 2.2 *Let $c^t = b^0 \times b^1 \times \ldots \times b^t$ and denote the length of c^t by l_t. Set $\mathcal{C} = \{N : 1 \le N \le t$ and b^N ends with a 1$\}$. If $|\mathcal{C}|$ is odd, then $\omega(-l_t, 2l_t) = \overline{c^t} + c^t$ and if $|\mathcal{C}|$ is even, then $\omega(-l_t, 2l_t) = c^t + c^t$.*

For the rest of the paper, $x = b^0 \times b^1 \times b^2 \ldots$ will denote a fixed nonperiodic recurrent sequence in X and ω and ν will be as above. As before, set $l_n = l([b^0 \times b^1 \times b^2 \ldots \times b^n])$. Set $M = \overline{O(\omega)}$, and S the shift on M. Since $\nu \in O_x$ and $O_x \subset \overline{O(\omega)}$, we have $\nu \in M$. Now take $\zeta \in M$. By the definition of M, for every $L \in \mathbf{N}$ we can find $n \in \mathbf{Z}$ such that $S^n(\omega)(-L, 2L+1) = \zeta(-L, 2L+1)$. Since x is not periodic, the symbol "1" appears somewhere in $b^t \times b^{t+1} \times \ldots$ for every t, and thus the dual of $\zeta(-L, 2L+1)$ must appear somewhere in ω. This shows that $\zeta \in M \Rightarrow \overline{\zeta} \in M$.

3 IP Cluster Points and Idempotents in $H(M)$

In this section , we show that when (M, S) is viewed as a **Z**-cascade, the IP cluster points and idempotents coincide and form a set of cardinality 4. One can prove this either by explicitly finding the IPCP's, or by computing $J(X)$, getting a finite set, and then using the fact that $\overline{J(X)} = I(X)$. Either method requires factoring M onto a simpler space, computing the set of idempotents there, and then working back up to (M, S). It is the second method which we shall adopt in this section as it is more elegant.

In order to factor M onto a simpler space, we first give a general definition of a factor of a dynamical system and then describe the particular systems we will use for our problem. This set up is motivated by [M].

A dynamical system (X, T_2) is said to be a factor of a dynamical system (Y, T_1) if there exists an onto continuous map $\pi : Y \to X$ such that $T_2 \circ \pi = \pi \circ T_1$. We also say that (Y, T_1) is an extension of (X, T_2). The extension is said to be almost one to one if π is one to one on $X - X'$ where X' is a subset of X of first category.

We construct a factor of (M, S) in two stages. First, define $Y = M/\{Id, \phi\}$ where $\phi : M \to M$ is defined by $\phi(\zeta) = \overline{\zeta}$, and let π_1 be the quotient map of this identification. Thus π_1 identifies dual elements of M. Denote

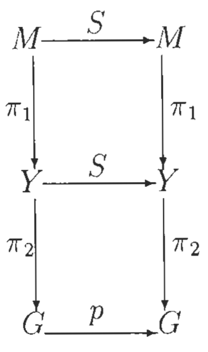
$$\pi_1(\omega) = \pi_1(\overline{\omega}) \text{ by } y_1 \text{ and } \pi_1(\nu) = \pi_1(\overline{\nu}) \text{ by } y_2.$$

Denote the projection of S onto Y again by S. Thus $S^n y_1 = \pi_1(S^n \omega) = \pi_1(S^n \overline{\omega})$ and $S^n y_2 = \pi_1(S^n \nu) = \pi_1(S^n \overline{\nu})$. y_1 and y_2 are then positively asymptotic. (M, S) is now an extension of (Y, S).

Next, set $G = \{(a_n)_{n=1}^{\infty} : 0 \leq a_n < l_n \text{ and } a_n = a_{n+1} (mod\, l_n)\}$. Let 1 denote the element $(1, 1, \ldots)$ in G and 0 the element $(0, 0, \ldots)$. Let $p : G \to G$ be defined by $p(g) = g + 1$ where addition is mod l_n in the n^{th} coordinate. (G, p) is a dynamical system and one can construct an almost one to one extension $\pi_2 : Y \to G$ as follows:

Note that if $\zeta = \lim_{k_i \to \infty} S^{k_i}(\omega)$ then for every n, $a_n = \lim_{i \to \infty}(k_i \pmod{l_n})$ exists [M, lemma 4b]. If in addition, $\overline{\zeta} = \lim_{k_j \to \infty} S^{k_j}(\omega)$, then by corollary 2.2 and [M, proposition 5], $\lim_{i \to \infty}(k_i \pmod{l_n}) = \lim_{j \to \infty}(k_j \pmod{l_n})$. So the map $\pi_2 :$ $Y \to G$ sending $y = \pi_1(\lim_{k_i \to \infty} S^{k_i}(\omega))$ to $(a_n)_{n=1}^{\infty}$ where $a_n = \lim_{i \to \infty}(k_i \pmod{l_n})$ is well defined, and by [M, proposition 5], π_2 is one to one off the orbit of 0 in G and two to one otherwise. So (Y, S) is an almost one to one extension of (G, p). In particular, we have $\pi_2(y_1) = \pi_2(y_2) = 0$ [M, proposition 5] and similarly, $\pi_2(S^n y_1) = \pi_2(S^n y_2) = p^n(0)$ for every $n \in \mathbf{Z}$.

This construction leads to the following commutative diagram:



Setting $\pi = \pi_2 \circ \pi_1$, we can summarize the preceding facts with the following properties:

1) $|\pi^{-1}(g)| = 2$ for every g not in $O(0)$, the orbit of 0 under p.
2) $|\pi^{-1}(g)| = 4$ for every $g \in O(0)$.
3) $\pi(\eta) = \pi(\overline{\eta})$ for every $\eta \in M$.
4) $\pi(S^n\omega) = \pi(S^n\nu) = p^n(0)$ for every $n \in \mathbf{Z}$.

We will now compute $I(M)$ and $J(M)$. We first compute

$I(M)$ in the $\mathbf{N}$-action case. We do this by finding $I(G)$ first and then working our way back to $I(M)$. The following three lemmas give us the necessary tools; the first one tells us how IP cluster points of a system relate to those of its extension.

Lemma 3.1 *Given an extension* (A, α) *of* (B, β) *with* (A, α) *and* (B, β) $\mathbf{N}$*-cascades,*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{\beta} & B \end{array}$$

IPCP's of A^A *project to IPCP's of* B^B*, i.e. If* $f \in A^A$ *is an IPCP along an IP set* P *and* $\pi \circ f = f' \circ \pi$ *then* f' *is an IPCP in* B^B *along* P*.*

Proof:

Let U' be a neighborhood of f' in B^B. Then we can find $k \in \mathbf{N}$, points $b_1, b_2, \ldots, b_k$ in B and open subsets $U'_1, U'_2 \ldots U'_k$ of B such that

$$f' \in \cap_{i=1}^{k} \{g \in B^B : g(b_i) \in U'_i\} \subseteq U'.$$

For each $i \in \{1, 2, \ldots, k\}$, pick $a_i \in A$ such that $\pi(a_i) = b_i$ and let $U_i = \pi^{-1}(U'_i)$, an open subset of A. Then $\pi(f(a_i)) = f'(\pi(a_i)) = f'(b_i) \in U'_i$, so $f(a_i) \in U_i$. Thus if $U = \cap_{i=1}^{k}\{g \in A^A : g(a_i) \in U_i\}$, then U is a neighborhood of f in A^A.

We are assuming f is an IPCP along P, thus $\{n \in P : \alpha^n \in U\}$ contains an IP set. However:

$$\begin{aligned} & \{n \in P : \alpha^n \in U\} \\ = & \{n \in P : \alpha^n(a_i) \in U_i\} \text{ for every } i \\ \subset & \{n \in P : \pi \circ \alpha^n(a_i) \in \pi(U_i)\} \text{ for every } i \\ = & \{n \in P : \beta^n \circ \pi(a_i) \in \pi(U_i)\} \text{ for every } i \\ = & \{n \in P : \beta^n(b_i) \in U'_i\} \text{ for every } i \\ & \{n \in P : \beta^n \in U'\}. \end{aligned}$$

Therefore $\{n \in P : \beta^n \in U'\}$ contains an IP set and f' is an IPCP in B^B along P. □

Lemma 3.2 *Let (X,T) be a metric $\mathbf{N}$-cascade. Suppose that X contains two positively asymptotic points x and y. Then $\theta \in H(X) \Rightarrow \theta(x) = \theta(y)$.*

Proof: We will prove this by contradiction. Assume $\theta(x) \neq \theta(y)$ and pick ϵ so that $d(\theta(x), \theta(y)) > 4\epsilon$. We can then find U and V, ϵ neighborhoods of $\theta(x)$ and $\theta(y)$ respectively, and define

$$W = \{g \in X^X : g(x) \in U\} \cap \{g \in X^X : g(y) \in V\}.$$

Notice that $\theta \in W$ and U and V are disjoint.

Since x and y are positively asymptotic, we can find N such that for every $n > N$, $d(T^n x, T^n y) < \epsilon$. This means that if $T^m(x) \in U$ and $m > N$, then $T^m(y) \notin V$. Thus

$$|\{n : T^n(x) \in U\} \cap \{n : T^n(y) \in V\}| < \infty,$$

which says that $\{n : T^n \in W\}$ is finite. But this contradicts θ being a limit point of $\{T^n\}$ and thus θ cannot belong to $H(X)$. □

Lemma 3.3 *For a minimal cascade (Y,S), if Y is not distal, then every minimal left ideal of $E(Y)$ contains more than one idempotent.*

Proof: (In this proof we use some basic facts from topological dynamics, for which we refer the reader to [A]. In [A], the author uses the notation xT to express a function T acting on an element x. We use $T(x)$. This explains why, in the proof below, minimal *left* ideals are used instead of [A]'s minimal *right* ideals.)

Recall that (Y,S) is minimal iff $Y \neq \emptyset$ and $Y = \overline{O(y)}$ for every $y \in Y$. This in turn implies that every $y \in Y$ is almost periodic, i.e. for every neighborhood U of y, the set of return times of y to U under the map S occurs with bounded gaps. (see [A] theorem 7, page 11).

Now let I be a minimal left ideal of $E(Y)$ and set J to be the set of idempotents in $I \cap H(I)$. Since every $y \in Y$ is almost periodic, for each $y \in Y$ there exists $u \in J$ such that $uy = y$ (see [A], theorem 12, page 88). If J contains only one element, $J = \{u\}$, then we have $uy = y$ for every $y \in Y$. So u is the identity on Y and $E(Y)$ contains no proper minimal left ideals. Clearly, the function $u \times u$ belongs to $H(Y \times Y)$ (if $S^{n_k} \to u$ then $(S \times S)^{n_k} \to u \times u$). Also, $E(Y \times Y)$ contains no proper minimal left ideals, since every idempotent $v \in E(Y \times Y)$ must be of the form $v_1 \times v_1$ where v_1 is an idempotent in $E(Y)$, implying that the identity on $Y \times Y$ is the only idempotent of $E(Y \times Y)$. But now $(u \times u)(x,y) = (x,y)$ for every $(x,y) \in Y \times Y$. This implies that every point in $Y \times Y$ is almost periodic (again see [A], theorem 12, page 88) and hence (Y,S) is distal (see [A], corollary 4, page 68). $\square$

A dynamical system (X,T) is said to be equicontinuous if given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $d(x,y) < \delta$, then $d(T^n x, T^n y) < \epsilon$ for every n. Furthermore (X,T) is equicontinuous if and only if X can be given a group structure which makes it a compact topological group, and there is an element $x_0 \in X$ such that $Tx = x_0 x$ for all $x \in X$ and such that the orbit of x_0 under the group action is dense in X [P, theorem 2.11, page 154]. Using this characterization, one can easily verify that the factor (G,p) of (M,S) is equicontinuous (if $\mathbf{l} = (l_1, l_2, \ldots)$ then G can be thought of as the group of $\mathbf{l}$-adic integers, equipped with the product topology, and the element $1 \in G$ plays the role of x_0). So (G,p) is equicontinuous, which implies that $E(G)$ is a group [A, theorem 3, page 52] . So $J(G) = \{Id_G\}$. Since $I(G) = \overline{J(G)}$, we have $I(G) = J(G) =$

$\{Id_G\}$.

By lemma 3.1, the only candidates for $I(Y)$ are the elements of $E(Y)$ which project to Id_G. In other words, we are looking for $f \in E(Y)$ such that $\pi_2 \circ f = Id_G \circ \pi_2$. Since π_2 is almost one to one, we know such an f must be the identity off the orbits of y_1 and y_2. On the orbits of y_1 and y_2, the only possibilities are Id_Y, f_1, f_2, f_3, where:

for each $n \in \mathbf{N} \cup \{0\}$, $f_1(S^n y_1) = f_1(S^n y_2) = S^n y_1$,
for each $n \in \mathbf{N} \cup \{0\}$, $f_2(S^n y_1) = f_2(S^n y_2) = S^n y_2$, and
for each $n \in \mathbf{N} \cup \{0\}$, $f_3(S^n y_1) = S^n y_2$ and
$f_3(S^n y_2) = S^n y_1$.

By lemma 3.2, f_3 and Id_Y are not in $H(Y)$ and thus not in $I(Y)$. Since $I(Y) \neq \emptyset$, one of f_1 or f_2 must be in $I(Y)$; since Y is not distal (it contains asymptotic points) by lemma 3.3, both f_1 and f_2 belong to $I(Y)$.

We now have $I(Y) = J(Y) = \{f_1, f_2\}$.

Moving up the tower (still in the $\mathbf{N}$-action case only), the candidates for $I(M)$ are the functions g which project to either f_1 or f_2. In other words, we are looking for $g \in E(M)$ such that $\pi_1 \circ g = f_1 \circ \pi_1$ or $\pi_1 \circ g = f_2 \circ \pi_1$. In the first situation, we must have $g(\{\omega, \overline{\omega}, \nu, \overline{\nu}\}) \subset \{\omega, \overline{\omega}\}$ and in the second, $g(\{\omega, \overline{\omega}, \nu, \overline{\nu}\}) \subset \{\nu, \overline{\nu}\}$. We are further restricted by the necessity that g must preserve duals, because if $g = \lim_{i\to\infty} S^{n_i}$, then $g(\overline{\zeta}) = \lim_{i\to\infty} S^{n_i}(\overline{\zeta}) = \lim_{i\to\infty}(S^{n_i}\overline{\zeta}) = \lim_{i\to\infty}(\overline{S^{n_i}\zeta}) = \overline{\lim_{i\to\infty}(S^{n_i}\zeta)} = \overline{g(\zeta)}$, with the second to last equality holding by the continuity of the shift map S. Thus the possible candidates for $I(M)$ can be divided into 8 classes, denoted by G_1 through G_8, and listed below according to their effect on $\omega, \overline{\omega}, \nu$ and $\overline{\nu}$ (note that every map in G_1 through G_8 commutes with the shift on the orbits of $\omega, \overline{\omega}, \nu$ or $\overline{\nu}$):

$$G_1 : \omega \leftrightarrow \overline{\omega}, \; \overline{\nu} \to \omega, \; \nu \to \overline{\omega}$$
$$G_2 : \nu \leftrightarrow \overline{\nu}, \; \omega \to \overline{\nu}, \; \overline{\omega} \to \nu$$
$$G_3 : \omega \leftrightarrow \overline{\omega}, \; \overline{\nu} \to \overline{\omega}, \; \nu \to \omega$$
$$G_4 : \nu \leftrightarrow \overline{\nu}, \; \omega \to \nu, \; \overline{\omega} \to \overline{\nu}$$

$$G_5: \omega \to \omega, \ \overline{\omega} \to \overline{\omega}, \overline{\nu} \to \omega, \ \nu \to \overline{\omega}$$
$$G_6: \omega \to \overline{\nu}, \ \overline{\omega} \to \nu, \overline{\nu} \to \overline{\nu}, \ \nu \to \nu$$
$$G_7: \omega \to \omega, \ \overline{\omega} \to \overline{\omega}, \overline{\nu} \to \overline{\omega}, \ \nu \to \omega$$
$$G_8: \omega \to \nu, \ \overline{\omega} \to \overline{\nu}, \overline{\nu} \to \overline{\nu}, \ \nu \to \nu$$

The only idempotents from the above list are the maps $g_i \in G_i, i = 5, 6, 7, 8$ which are the identity off the orbits of $\omega, \overline{\omega}, \nu$ or $\overline{\nu}$. Thus $J(M)$ is finite and since $I(M) = \overline{J(M)}$, we have $I(M) = J(M)$. Now by lemma 3.2 (still in the **N**-action case), g_5 and g_6 are not in $H(M)$ since $g_i(\omega) \neq g_i(\nu)$ for $i = 5, 6$. By lemma 3.3, $J(M) = \{g_7, g_8\}$. And so, $I(M) = \{g_7, g_8\}$.

The above computation was for (M, S) viewed as an **N**-action. To extend to the **Z**-action case, we make use of the following elementary proposition.

Proposition 3.4 *Let P be an IP subset of **Z**, generated by $\{p_n\}_{n=1}^{\infty}$. If p_n is positive for an infinite number of n, we denote by P^+ the IP set generated by the positive p_n's. If p_n is negative for an infinite number of n, we denote by P^- the IP set generated by the negative p_n's. Then f is an IPCP for a **Z**-action cascade along an IP set P iff f is an IPCP for at least one of the corresponding $\mathbf{Z}^+$ or $\mathbf{Z}^-$ actions, along P^+ or P^- respectively. ($\mathbf{Z}^+$ stands for **N**).*

Now the arguments used to compute $I(M)$ for the **N**-action case can be repeated to yield that for the $\mathbf{Z}^-$-action case, $I(M) = J(M) = \{g_5, g_6\}$ (negative instead of positive asymptoticity accounts for the difference in the final result). We leave the details of the computation in the $\mathbf{Z}^-$ case to the reader.

Altogether, we have proven the following theorem:

Theorem 3.5 *The dynamical system (M, S) has $I(M) = J(M) = \{g_5, g_6, g_7, g_8\}$. In particular, the idempotents in the enveloping semigroup exactly correspond to the IP cluster points.*

Our theorem produces the following corollaries:

Corollary 3.6 *Let g be a map in $E(M)$ such that $g(\omega) \in \{\overline{\omega}, \overline{\nu}\}$. Then there exists a neighborhood U of g such that $\{n \in \mathbf{N} : S^n \in U\}$ does not contain an IP set.*

Corollary 3.7 *Let g be a map in $E(M)$ such that $g(\omega) \in \{\omega, \nu\}$ and such that g interchanges η and $\overline{\eta}$ for some $\eta \notin O(\omega) \cup O(\nu) \cup O(\overline{\omega}) \cup O(\overline{\nu})$. Then there exists a neighborhood U of g such that $\{n \in \mathbf{N} : S^n \in U\}$ does not contain an IP set.*

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California State University at Bakersfield
Bakersfiled, CA 93311
e-mail address: khaddad@ultrix6.cs.csubak.edu

Swarthmore College
Swarthmore, PA 19081
e-mail address: aimee@swarthmore.edu