

Commuting endomorphisms of the circle

AIMEE S. A. JOHNSON

Mathematics Department, Tufts University, Medford, MA 02155, USA

DANIEL J. RUDOLPH

Department of Mathematics, University of Maryland, College Park, MD 20742, USA

(Received 13 November 1990)

Abstract. In this paper the results of Shub and Sacksteder are extended to the following theorem: let f_1 and f_2 be two commuting, expansive, orientation-preserving maps of the circle with a common fixed point and with both in C^{1+r} or C^r , $r \geq 2$. Assume f_1 is p -to-1 and f_2 is q -to-1 where p and q generate a nonlacunary semigroup. Then there exists a diffeomorphism g of the same class such that $gf_1g^{-1} = T_p$ and $gf_2g^{-1} = T_q$.

1. Introduction

Consider the circle to be the group $[0, 1)$ under addition modulo 1. By T_p we will mean the function $T_p(x) = px \bmod 1$. This paper extends the following results.

THEOREM. [Sh]. Let $f: S^1 \rightarrow S^1$ be a C^1 function such that $|df| > 1$ on all of S^1 , where df is the derivative of f . If f is a p -to-1 map then f is homeomorphically conjugate to T_p , i.e. there exists a homeomorphism g with $g^{-1}fg = T_p$.

THEOREM. [S]. Let f_1 and f_2 be expanding, orientation-preserving maps of S^1 that commute, have degrees n and m respectively, and are of class C^{n-} ($n \geq 2$). Then if n and m are relatively prime, there is a diffeomorphism g of class C^{n-} satisfying $g(f_1(x)) = T_n g(x)$ and $g(f_2(x)) = T_m g(x)$.

There is a well known obstruction to Shub's homeomorphism g being a diffeomorphism. We describe this in § 2. However, we can now extend Sacksteder's result to integers p and q where p and q generate a nonlacunary semigroup of \mathbb{N} . In § 3 this theorem is stated formally along with the definition of nonlacunary. §§ 4 and 5 contain the results needed to prove this theorem. In § 6 we will discuss the case when f_1 and f_2 do not have a common fixed point.

2. The diffeomorphism obstruction

From Shub's theorem we know $f = gT_p g^{-1}$. So g must send the fixed points of T_p to those of f . Let us write this as $g(i/(p-1)) = z_i$. If g is differentiable then

$$df(x) = dg(T_p g^{-1}(x)) dT_p(g^{-1}(x)) dg^{-1}(x).$$

In particular,

$$df(z_i) = dg(i/(p-1)) dT_p(i/(p-1)) dg^{-1}(z_i).$$

But $dg(i/(p-1)) \times dg^{-1}(z_i) = 1$, so

$$df(z_i) = dT_p(i/p-1) = p.$$

Thus g cannot be differentiable if f has a derivative at a fixed point not equal to p . Applying this idea to powers of f gives obstructions from all periodic orbits of f . According to Katok [Ka] these are the only obstructions, and an alternative proof of our main theorem can be constructed along these lines.

3. Main theorem

Definition. A semigroup of \mathbb{N} is said to be nonlacunary if it is not contained in a singly generated semigroup. For instance, the multiplicative semigroup generated by 6 and 10 is nonlacunary, the one generated by 4 and 8 is not.

Recall that $f \in C^{1+\epsilon}$ if f has a continuous derivative that is Hölder, with Hölder constant ϵ . We say $f \in C^r$, $r \geq 2$, if f has r continuous derivatives. By \mathcal{C} let us mean one of these classes of functions. If $\mathcal{C} = C^{1+\epsilon}$ then by \mathcal{C}' we mean C^ϵ . If $\mathcal{C} = C^r$ then $\mathcal{C}' = C^{r-1}$.

MAIN THEOREM *Let f_1 and f_2 be two commuting, orientation-preserving maps in \mathcal{C} with a common fixed point. Assume they are both expanding and that f_1 is p -to-1 and f_2 is q -to-1 where p and q generate a nonlacunary semigroup. Then there exists a diffeomorphism $g \in \mathcal{C}$ such that $gf_1g^{-1} = T_p$ and $gf_2g^{-1} = T_q$.*

PROPOSITION 1. *For f_1 and f_2 as in the theorem, there exists a homeomorphism g that conjugates f_1 to T_p and f_2 to T_q .*

Proof. Let 0 be the common fixed point. From Shub there is a homeomorphism $g : S^1 \rightarrow S^1$ with $g^{-1}f_1g = T_p$. Since T_p and f_1 both have 0 as a fixed point we can assume g fixes 0 (compose g with a rotation as necessary). Then $g^{-1}f_2g = \tilde{f}_2$ is a q -to-1 orientation-preserving map of S^1 to itself, fixing 0 and commuting with T_p .

Let \hat{f}_2 be the unique lift of \tilde{f}_2 to a homeomorphism of the universal cover, \mathbb{R} , fixing 0. Thus $\hat{f}_2(px) = p\hat{f}_2(x)$ and for $n \in \mathbb{N}$, $\hat{f}_2(n) = qn$. It is then easy to show that $\hat{f}_2(\sum a_j/p^j) = q(\sum a_j/p^j)$ and hence $\hat{f}_2(x) = qx$, thus $\tilde{f}_2 = T_q$.

We want to show that this conjugation g is in \mathcal{C} . This will be done in the next two sections. First we generalize our theorem to the non-orientation preserving case.

COROLLARY OF MAIN THEOREM. *Let f_1 and f_2 be as described in the main theorem except not necessarily orientation preserving. Then there exists $g \in \mathcal{C}$ such that $gf_1g^{-1} = T_r$ and $gf_2g^{-1} = T_s$ with $|r| = p$ and $|s| = q$. The sign of r and s is determined by the orientation of the associated map.*

Proof. The set of continuous, monotone maps that commute with some T_m is $\{T_n + k/(m-1) : k = 0, \dots, m-2 \text{ and } n \in \mathbb{Z}\}$. This follows from an argument similar to the proof of Proposition 1 and the fact that any map that commutes with T_m can only permute its fixed points. Now we use the main theorem to find $g \in \mathcal{C}$ such that $gf_1g^{-1} = T_{p^2}$ and $gf_2g^{-1} = T_{q^2}$. As before, we can assume g fixes 0. Then gf_1g^{-1} commutes with T_{p^2} so must have the form $T_n + k/(p^2-1)$. Since it fixes zero and has degree p it must equal T_r with $|r| = p$. Similarly for gf_2g^{-1} . \square

4. Preliminary results

In this section we will define a measure μ and show that it is f_1 and f_2 invariant. Using the structure of Parry and Pollicott [PP] and Krzyzewski [Kr] on the operator introduced by Ruelle [R], define $L: C(S^1) \rightarrow C(S^1)$ by $L(w)(x) = \sum_{y: f_1 y = x} w(y) / df_1(y)$. In Parry's notation, this is $L_{(-\log df_1)}$. If we let \tilde{f}_1 be the lift of f_1 so that $\tilde{f}_1: [0, 1) \rightarrow [0, p)$ is 1-1 then we can rewrite this operator as

$$L(w)(x) = \sum_{i=0}^{p-1} \frac{w(\tilde{f}_1^{-1}(x+i))}{df_1(\tilde{f}_1^{-1}(x+i))}$$

From [Kr] we get the following.

PROPOSITION 2. *There exists a strictly positive eigenfunction $h \in \mathcal{C}'$, corresponding to eigenvalue 1, of the operator L .*

Thus we have $L(h) = h$.

PROPOSITION 3.

$$\int_A h(x) dx = \int_{\tilde{f}_1^{-1}A} h(x) dx$$

Proof. We have

$$\begin{aligned} \int_A h(x) dx &= \int_A Lh(x) dx = \int_A \sum_{i=0}^{p-1} \frac{h(\tilde{f}_1^{-1}(x+i))}{df_1(\tilde{f}_1^{-1}(x+i))} dx \\ &= \int_{A \cup A+1 \cup \dots \cup A+(p-1)} \frac{h(\tilde{f}_1^{-1}(z))}{df_1(\tilde{f}_1^{-1}(z))} dz \end{aligned}$$

Now use the change of variables given by $y = \tilde{f}_1^{-1}(z)$. Then $dy = dz / df_1(\tilde{f}_1^{-1}(z))$ and we get $\int_{\tilde{f}_1^{-1}A} h(y) dy$ as wanted. \square

Definition 4. Take μ to be the measure given by $d\mu = h(x) dx$.

By Proposition 3, μ is f_1 invariant. We next want to show that it is also f_2 invariant.

PROPOSITION 5. (See [Kr].) *The system (S^1, f_1, μ) is exact. In particular, it is ergodic and $h_\mu(f_1) > 0$.*

Let $f_2^* \mu$ be the measure defined by $f_2^* \mu(E) = \mu(f_2^{-1}E)$. Then $f_2^* \mu$ is f_1 invariant, since f_1 and f_2 commute.

PROPOSITION 6. $f_2^* \mu = \mu$.

Proof. As f_2 is differentiable,

$$\frac{df_2^* \mu}{d\mu}(x) = \sum_{y: f_2(y)=x} \frac{h(y)}{h(x) df_2(y)} > 0$$

Since μ is ergodic for f_1 , this gives us the result. \square

This tells us that μ is invariant for f_2 .

5. Completion of result

Consider $g^* \mu$, where g is the conjugation from Proposition 1. Since μ is f_1 and f_2 invariant, $g^* \mu$ is T_p and T_q invariant.

PROPOSITION 7. $g^*\mu$ is Lebesgue measure λ .

Proof. By Proposition 5 we know that (S^1, f_1, μ) has positive entropy. Thus $(S^1, T_p, g^*\mu)$ also has positive entropy. Using the result from [J], we know the only measures that are invariant and ergodic for T_p and T_q are λ and measures of entropy zero. \square

Proof of main theorem. Rewriting this we get $(g^*)^{-1}\lambda = \mu$. By construction μ is absolutely continuous with respect to Lebesgue measure. We have $(g^*)^{-1}\lambda(A) = \int_A h(x) dx$. Now let $A = [0, \tau]$, then $(g^*)^{-1}\lambda(A) = \lambda(gA) = \lambda[0, g(\tau)] = g(\tau)$, because g is an increasing function. So we have $g(\tau) = \int_0^\tau h(x) dx$. By the fundamental theorem of calculus $dg(\tau) = h(\tau)$ thus since $h \in \mathcal{C}'$, we have $g \in \mathcal{C}$. \square

6. If there is no common fixed point

In the construction of the conjugation g in Proposition 1, we used the existence of a common fixed point for f_1 and f_2 . Now we want to discuss the situation when no such common fixed point exists, eventually classifying the possible conjugations.

THEOREM 8. Let f_1 and f_2 be two commuting, expanding, orientation-preserving maps in \mathcal{C} such that f_1 is p -to-1 and f_2 is q -to-1 where p and q generate a nonlacunary semigroup. Then there exists a diffeomorphism $g \in \mathcal{C}$ such that $gf_1g^{-1} = T_p$ and $gf_2g^{-1} = T_q + i/(p-1)$ for some i .

Proof. Since f_2 is q -to-1, it must have $q-1$ fixed points. By the commutivity of f_1 and f_2 , f_1 can only permute these. Thus there is some power k of f_1 that has a common fixed point with f_2 and we can apply our theorem to find a conjugation $\varphi \in \mathcal{C}$ with

$$\varphi^{-1}f_2\varphi = T_q \quad \text{and} \quad \varphi^{-1}f_1^k\varphi = T_p^k. \tag{1}$$

Let $\bar{f}_1 = \varphi^{-1}f_1\varphi$. Then \bar{f}_1 is a p -to-1 map, T_q is a q -to-1 map, and we can repeat the above process to find an integer n such that \bar{f}_1 and T_q^n have a common fixed point. Thus there is a conjugation $\psi \in \mathcal{C}$ with

$$\psi^{-1}\bar{f}_1\psi = T_p \quad \text{and} \quad \psi^{-1}T_q^n\psi = T_q^n. \tag{2}$$

Define a measure $\psi^*\lambda$ by $\psi^*\lambda(A) = \lambda(\psi^{-1}A)$, where λ is Lebesgue measure. Because ψ is smooth we can use an argument much like that in Proposition 6 to show $\psi^*\lambda = \lambda$. But the only continuous, orientation preserving, 1-1 maps that leave Lebesgue measure invariant are rotations. So we will write ψ as R_a .

Using (1) and (2) we have

$$R_a^{-1}\varphi^{-1}f_1\varphi R_a = T_p \quad \text{and} \quad R_a^{-1}\varphi^{-1}f_2\varphi R_a = R_a^{-1}T_q R_a.$$

T_p and $R_a^{-1}T_q R_a$ must commute since f_1 and f_2 do. But $T_p R_a^{-1}T_q R_a(x) = T_p(qx + qa - a) = pqx + pqa - pa$, and $R_a^{-1}T_q R_a T_p(x) = R_a^{-1}T_q R_a(px) = qpx + qa - a$. Thus we get $pqa - pa = qa - a$, or $p(qa - a) = qa - a$ which says $qa - a$ is a fixed point of T_p . That means $qa - a = i/(p-1)$ for some $i = 0, \dots, p-2$ and thus $a = i/(p-1)(q-1)$.

We can then rewrite $R_a^{-1}T_q R_a(x) = qx + qa - a = qx + a(q-1) = qx + i/(p-1)$. Putting $g = R_a^{-1}\varphi^{-1}$ gives the result of the theorem. \square

PROPOSITION 9. *If $(p-1)$ and $(q-1)$ are relatively prime then f_1 and f_2 always have a common fixed point. So Theorem 8 in this particular case reduces to the statement of the main theorem.*

Proof. In the proof of Theorem 8 we showed $R_a^{-1}\varphi^{-1}f_1\varphi R_a = T_p$ and $R_a^{-1}\varphi^{-1}f_2\varphi R_a = R_a^{-1}T_qR_a$. So f_1 and f_2 have a common fixed point iff T_p and $R_a^{-1}T_qR_a$ do. If $p-1$ and $q-1$ are relatively prime we can rewrite $a = i/(p-1)(q-1)$ as $i_1/(p-1) + i_2/(q-1)$, and R_a as $R_{a_2}R_{a_1}$. But then we have f_1 conjugate to $R_{a_1}T_pR_{a_1}^{-1}$ and f_2 conjugate to $R_{a_2}^{-1}T_qR_{a_2}$, both conjugations by φR_{a_2} . Notice that both these maps take 0 to 0.

Thus f_1 and f_2 have a common fixed point and we can assume that the conjugation in Theorem 8 sends it to 0. Then Theorem 8 says $gf_2g^{-1}(0) = (T_q + i/(p-1))(0) = 0$ which shows that gf_2g^{-1} in fact must equal T_q . Thus Theorem 8 reduces to the statement of the main theorem in this special case. \square

We have

$$a \in G = \left\{ \frac{l}{(p-1)(q-1)} \pmod 1 : l \in \mathbb{Z} \right\}.$$

If $(p-1)$ and $(q-1)$ are relatively prime then this group is the same as

$$H = \left\{ \frac{i}{p-1} + \frac{j}{q-1} \pmod 1 : i, j \in \mathbb{Z} \right\}$$

and the pair f_1 and f_2 is only conjugate to T_p and T_q . In general these two groups are not the same and we get the following:

THEOREM 10. *Let f_1 and f_2 be two commuting, expanding, orientation-preserving maps in \mathcal{C} with f_1 p -to-1 and f_2 q -to-1 where p and q generate a nonlacunary semigroup. Then there exists $g \in \mathcal{C}$ that conjugates the pair to T_p and $R_a^{-1}T_qR_a$ where $a \in G$ is unique up to its coset aH . Thus the number of possible classes for pairs f_1, f_2 is $|G|/|H|$.*

Proof. We need to first show that if f_1, f_2 is conjugate to $T_p, R_{a_1}^{-1}T_qR_{a_1}$, then they are also conjugate to $T_p, R_{a_2}^{-1}T_qR_{a_2}$ where a_2 is an arbitrary element in a_1H . So we will show $T_p, R_{a_1}^{-1}T_qR_{a_1}$ is conjugate to $T_p, R_{a_2}^{-1}T_qR_{a_2}$. Let $a_1 = i_1/(p-1)(q-1)$ and $a_2 = i_2/(p-1)(q-1)$. Then i_1 and i_2 are such that

$$\frac{i_1}{(p-1)(q-1)} = \frac{i_2}{(p-1)(q-1)} + \frac{c_1}{p-1} + \frac{c_2}{q-1}$$

for some c_1 and c_2 and we can write

$$R_{a_1} = R_{a_2}R_{c_2/(q-1)}R_{c_1/(p-1)}.$$

Using $R_{c_1/(p-1)}^{-1}$ as the conjugation we have $T_p, R_{a_1}^{-1}T_pR_{a_1}$ conjugate to

$$R_{c_1/(p-1)}^{-1}T_pR_{c_1/(p-1)}, \quad R_{c_2/(q-1)}^{-1}R_{a_2}^{-1}T_qR_{a_2}R_{c_2/(q-1)}.$$

But the first is equal to T_p and the second to $R_{a_2}^{-1}T_qR_{a_2}$. This gives the result.

Next we need to show that if f_1, f_2 is conjugate to $T_p, R_{a_1}^{-1}T_pR_{a_1}$ and to $T_p, R_{a_2}^{-1}T_qR_{a_2}$ then a_1 and a_2 must be in the same coset. So here we have ψ such that

$\psi^{-1}T_p\psi = T_p$ and $\psi^{-1}R_{a_2}^{-1}T_qR_{a_2}\psi = R_{a_1}^{-1}T_qR_{a_1}$. From the first relationship we see that ψ must have the form $R_{c/(p-1)}$. From the second equation we see that

$$\frac{i_2}{(p-1)(q-1)} + \frac{c}{p-1} = \frac{i_1}{(p-1)(q-1)},$$

which gives the result. \square

REFERENCES

[J] Aimee S. A. Johnson, Measures on the circle invariant for a nonlacunary subsemigroup of the integers, *Israel J. to appear*.
 [Ka] A. Katok. Private correspondence.
 [Kr] K. Krzyzewski. Some results on expanding mappings. *Astérisque* 50 (1977), 205-219.
 [PP] W. Parry & M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque*, to appear.
 [R] D. Ruelle. Thermodynamic formalism. *Encyclopedia of Mathematics and its Applications* 5. Addison-Wesley, 1978.
 [S] R. Sacksteder. Abelian semi-groups of expanding maps. *Springer Lecture Notes in Mathematics* 318. Springer, New York, 1972, pp 235-238.
 [Sh] M. Shub. Endomorphisms of compact differentiable manifolds. *Am. J. Math.* 91 (1969), 175-199.

$$\left\{ \begin{array}{l} \exists i \in \mathbb{Z} \\ \text{mod } i: i \equiv 1 \end{array} \right\} = H$$

and the pair λ and λ' is only conjugate to T_p and T_q in general these two groups are not the same and we get the following:

Theorem 10. Let λ and λ' be two commuting, expanding, orientation-preserving maps in \mathbb{F} with λ p -to-1 and λ' q -to-1 where p and q generate a nonlacunary semigroup. Then there exist $g \in \mathbb{F}$ that conjugates the pair to T_p and T_q where $a \in \mathbb{Q}$ is unique up to its coset aH . Thus the number of possible classes for pairs (λ, λ') is $|\mathbb{Q}/H|$. Proof. We need to first show that if λ, λ' is conjugate to T_p, T_q then they are also conjugate to $T_p, R_{a_1}^{-1}T_qR_{a_1}$ where a_1 is an arbitrary element in aH . So we will show $T_p, R_{a_1}^{-1}T_qR_{a_1}$ is conjugate to T_p, T_q . Let $\psi = i/(p-1)(q-1)$ and $\psi^{-1}T_p\psi = T_p$. Then ψ and ψ' are such that

$$\frac{i_1}{(p-1)(q-1)} + \frac{c}{p-1} = \frac{i_2}{(p-1)(q-1)} + \frac{c'}{p-1}$$

for some c and c' and we can write

$$R_{a_1}^{-1}T_qR_{a_1} = R_{a_2}^{-1}T_qR_{a_2}$$

Using $R_{a_1}^{-1}T_qR_{a_1}$ as the conjugation we have $T_p, R_{a_1}^{-1}T_qR_{a_1}$ conjugate to $R_{a_1}^{-1}T_pR_{a_1}, R_{a_1}^{-1}T_qR_{a_1}$. But the first is equal to T_p and the second to $R_{a_1}^{-1}T_qR_{a_1}$. This gives the result. Next we need to show that if λ, λ' is conjugate to $T_p, R_{a_1}^{-1}T_qR_{a_1}$ and to T_p, T_q then a_1 and a_2 must be in the same coset. So here we have ψ such that